Section 4.1 – Exponential Functions

• These functions model rapid growth or decay:
  – # of users on the Internet
    • 16 million (1995) → 957 million (late 2005)
  – Compound interest
  – Population growth or decline
Comparison

- Linear Functions
  - Rate of change is constant
- Exponential Functions
  - Change at a constant percent rate.
The Exponential Function

\[ f(x) = b^x \]

\( b \) is the base:

- It must be greater than 0, \( b > 0 \)
- It cannot equal 1, \( b \neq 1 \)

\( x \) can be any real number

The **domain** is the set of all real numbers, while the **range** is the set of all positive real numbers.

- Which of the following are exponential functions?
  
  \[ y = 3^x \quad \text{yes} \quad y = x^3 \quad \text{no} \]
  
  \[ y = 2(7)^x \quad \text{yes} \quad y = 2(-7)^x \quad \text{no} \]
Properties of Exponential Functions

1. \( b^m \cdot b^n = b^{m+n} \)
2. \( b^m / b^n = b^{m-n} \)
3. \( (b^m)^n = b^{m \cdot n} \)
4. \( (b \cdot c)^m = b^m \cdot c^m \)
5. \( (b / c)^m = b^m / c^m \)
6. \( b^{-m} = 1 / b^m \)

Example 1: \( f(x) = 3^x + 3^x \)

\[
f(x) = 2 \cdot 3^x \quad \text{or} \quad 2 \left( 3^x \right)
\]

Example 2: \( f(x) = \left( 3^x \right) \left( 3^4 \right) \left( 3^x \right) \)

\[
f(x) = 2^{2x+4}
\]
Simplify the following functions.

**Example 3:** \( f(x) = \frac{2^x}{2^4} = 2^{x-4} \)

**Example 4:** \( f(x) = \frac{1}{2^x} = 2^{-x} \)

**Example 5:** \( f(x) = (2^x)^4 = 2^{4x} \)

**Example 6:** \( f(x) = a^x + a^x + a^x = 3a^x \)

**Example 7:** \( f(x) = (a^x)(a^y)(a^z) = a^{x+y+z} \)

**Example 8:** \( f(x) = \frac{a^x}{a^y} = a^{x-y} \)

**Example 9:** \( f(x) = (a^3)^4 = a^{12} \)

**Example 10:** \( f(x) = 4^x + 3^x + 4^x = 2(4^x) + 3^x \)

**Example 11:**

\[
f(x) = 5^x + 3^x + 5^x + 3^x + 3^w
= 2(5^x) + 2(3^x) + 3^w
\]

**Example 12:**

\[
f(x) = 4^x \cdot 2^{5x+1}
= (2^2)^x \cdot 2^{5x+1}
= 2^{2x} \cdot 2^{5x+1}
\]

**Example 13:**

\[
f(x) = 2^{5x+1} \cdot 2^{3-2x}
= 2^{5x+1+3-2x} = 2^{3x+4}
\]
Example 3. Simplify the following expressions

(a) \(4^{x+6} \cdot 8^{2-x}\)

Solution

\[
4^{x+6} \cdot 8^{2-x} = (2^2)^{x+6} \cdot (2^3)^{2-x} = 2^{2(x+6)} \cdot 2^{3(2-x)} \\
= 2^{2x+12} \cdot 2^{6-3x} = 2^{(2x+12)+(6-3x)} = 2^{-x+18}.
\]

(b) \(\frac{27^{2x-3}}{9x-4}\)

Solution

\[
\frac{27^{2x-3}}{9x-4} = \frac{(3^3)^{2x-3}}{(3^2)^{x-4}} = \frac{3^{6x-9}}{3^{2x-8}} = 3^{(6x-9)-(2x-8)} = 3^{4x-1}.
\]

Exercise 3. Simplify the following expressions (i.e., reduce to a single exponential function).

(a) \(5^{x-2} \cdot 25^{3-x}\)

(b) \(3^{x-1} \cdot 9^{x-2} \cdot 27^{x-3}\)

(c) \(\frac{8^{x+4}}{16^{x-2}}\).
Example: Solve $4^{x-2} = 64^x$

$4^{x-2} = (4^3)^x$

$$64 = 4^3$$

$4^{x-2} = 4^{3x}$

If $b^M = b^N$, then $M = N$

$x-2 = 3x$

$-2 = 2x$

$-1 = x$

If the bases are already $=,$ just solve the exponents

Example: Solve $27^{x+3} = 9^{x-1}$

$$\left(3^3\right)^{x+3} = \left(3^2\right)^{x-1}$$

$3^{3x+9} = 3^{2x-2}$

$3x + 9 = 2x - 2$

$x + 9 = -2$

$x = -11$
Example 4. Solve each of the following equations for $x$.

(a) $4^x = 16^{2x-2}$

(b) $2^{3x} = 0.25$

(c) $3^{2x} - 6 \cdot 3^x - 27 = 0$.

Solution

(a) Simplify so that both sides have the same basis:

$4^x = 16^{2x-2}$

$4^x = (4^2)^{2x-2}$

$4^x = 4^{4x-4}$

It follows that $4x - 4 = x$ and $x = 4/3$.

(b) $2^{3x} = 0.25 = \frac{1}{4} = \frac{1}{2^2} = 2^{-2}$. Thus, $x^3 = -2$ and so $x = \sqrt[3]{-2}$.

(c) The idea lies in the fact that $3^{2x} = (3^x)^2$; this implies that the given equation is a quadratic equation in $3^x$. Let $y = 3^x$; then $3^{2x} - 6 \cdot 3^x - 27 = 0$ reads $y^2 - 6y - 27 = 0$. From

$$y^2 - 6y - 27 = (y + 3)(y - 9) = 0,$$

we conclude that $3^x = y = -3$ or $3^x = y = 9$.

Since $3^x > 0$, the equation $3^x = -3$ has no solutions. From $3^x = 9$, we get $x = 2$. Thus, the only solution is $x = 2$. 

Exercise 4. Solve each of the following equations for $x$.

(a) \(0.5^x = 0.125\)  
(b) \(3^x(3^x - 3) = 0\)  
(c) \(2^{2x} - 5 \cdot 2^x + 4 = 0\).
EXAMPLE

Solve the following equation for \( x \).

\[(2 - 3x)5^x + 4 \cdot 5^x = 0\]

SOLUTION

\[(2 - 3x)5^x + 4 \cdot 5^x = 0\]

This is the given equation.

Factor.

\[5^x [2 - 3x + 4] = 0\]

Simplify.

\[5^x [6 - 3x] = 0\]

Since \( 5^x \) and \( 6 - 3x \) are being multiplied, set each factor equal to zero.

\[5^x = 0 \quad 6 - 3x = 0\]

\[5^x \neq 0 \quad 2 = x\]

\[5^x \neq 0.\]
Example – Solve the following equation for \( x \)

a) \( 3^{2x+1} - 5 = 4 \)

1. Isolate the exponential expression and rewrite the constant in terms of the same base.

\[ 3^{2x+1} = 9 \]
\[ 3^{2x+1} = 3^2 \]
\[ 2x + 1 = 2 \]
\[ 2x = 1 \]
\[ x = \frac{1}{2} \]

b) \( 3^{x+1} = 9^{x-2} \)

1. Isolate the exponential expressions on either side of the \( = \). Here the we had to rewrite the 2nd expression in terms of the same base as the first.

\[ 3^{x+1} = (3^2)^{x-2} \]
\[ 3^{x+1} = 3^{2x-4} \]
\[ x + 1 = 2x - 4 \]
\[ x = 5 \]

2. Set the exponents equal to each other (drop the bases) and solve the resulting equation.
Graph Exponential Functions

When \((b > 1)\)

1) Graph \(y = 2^x\) for \(x = -3\) to 3

<table>
<thead>
<tr>
<th>(x)</th>
<th>(f(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>(2^{-3} = \frac{1}{8})</td>
</tr>
<tr>
<td>-2</td>
<td>(2^{-2} = \frac{1}{4})</td>
</tr>
<tr>
<td>-1</td>
<td>(2^{-1} = \frac{1}{2})</td>
</tr>
<tr>
<td>0</td>
<td>(2^0 = 1)</td>
</tr>
<tr>
<td>1</td>
<td>(2^1 = 2)</td>
</tr>
<tr>
<td>2</td>
<td>(2^2 = 4)</td>
</tr>
<tr>
<td>3</td>
<td>(2^3 = 8)</td>
</tr>
</tbody>
</table>
Graph Exponential Functions

When \(0 < b < 1\)

- Graph \(y = (1/2)^x\) for \(x = -3\) to 3

<table>
<thead>
<tr>
<th>(x)</th>
<th>(f(x))</th>
<th>(Y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>((1/2)^{-3} = 2^3 = 8)</td>
<td>8</td>
</tr>
<tr>
<td>-2</td>
<td>((1/2)^{-2} = 2^2 = 4)</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>((1/2)^{-1} = 2^1 = 2)</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>((1/2)^0 = 1)</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>((1/2)^1 = \frac{1}{2})</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>2</td>
<td>((1/2)^2 = \frac{1}{4})</td>
<td>(\frac{1}{4})</td>
</tr>
<tr>
<td>3</td>
<td>((1/2)^3 = \frac{1}{8})</td>
<td>(\frac{1}{8})</td>
</tr>
</tbody>
</table>
Characteristics of the graphs $f(x) = b^x$
of

• $x$ can be any value

• The resulting $y$ value will always be positive.

• The $y$-intercept is always (0,1)

• When $b > 1$, as $x$ increases, $y$ increases.

• When $0 < b < 1$, as $x$ increases, $y$ decreases.

• All graphs will pass through (0,1) ($y$ intercept)

• There are no $x$ intercepts.
Exponential Function with Base $e$

The number

$$e \approx 2.718281828459$$

where the approximation is given correct to 12 decimal places, is most useful for a base in an exponential function. The exponential function with base $e$ is called the natural exponential function.

The function $f(x) = e^x$ is called the natural exponential function.

**EXAMPLE**

$$\left( e^x \right) \left( e^{2x} \right) = e^{x+2x} = e^{3x}$$