Chapter 1. Quadratic Inequality (Lecture 7)
Solving a Quadratic Inequality

Given \[ f(x) = ax^2 + bx + c \]

Find the roots by equating to zero, \( ax^2 + bx + c = 0 \)

Solve by using either Factoring or Quadratic formula

Suppose \( X_1 \) and \( X_2 \) are the roots

Suppose \( X_1 = X_2 \) are the roots
Solving Quadratic Inequalities: 1st Method

Ex: Solve $x^2 + 3x - 10 < 0$

Solution:

$x^2 + 3x - 10 = 0$

$\Rightarrow (x + 5)(x - 2) = 0$

$\Rightarrow x = -5, \ x = 2$

$f(x) = x^2 + 3x - 10 \Rightarrow a = 1 > 0$

<table>
<thead>
<tr>
<th>x</th>
<th>$-\infty$</th>
<th>$-5$</th>
<th>$2$</th>
<th>$+\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2 + 3x - 10$</td>
<td>$+$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

Solution set $= (-5, 2)$
Solving Quadratic Inequalities: 2\textsuperscript{nd} Method

Solve $x(x - 1)(x + 4) \leq 0$

Solution: If $f(x) = x(x - 1)(x + 4)$, then $f$ is a polynomial function and is continuous everywhere. The zeros of $f$ are 0, 1, and $-4$,

$$(-\infty, -4) \quad (-4, 0) \quad (0, 1) \quad (1, \infty)$$

Now, at a test point in each interval, we find the sign of $f(x)$:

$$f(-5) = (-)(-)(-) = - \quad \text{so } f(x) < 0 \text{ on } (-\infty, -4)$$
$$f(-2) = (-)(-)(+) = + \quad \text{so } f(x) > 0 \text{ on } (-4, 0)$$
$$f\left(\frac{1}{2}\right) = (+)(-)(+) = - \quad \text{so } f(x) < 0 \text{ on } (0, 1)$$
$$f(2) = (+)(+)(+) = + \quad \text{so } f(x) > 0 \text{ on } (1, \infty)$$

Figure shows the sign chart for $f(x)$. Thus, $x(x - 1)(x + 4) \leq 0$ on $(-\infty, -4]$ and $[0, 1]$. 
Example: Consider the inequality \( x^2 - x - 6 > 0 \)

Solution: We can find the values where the quadratic equals zero by solving the equation,

\[
x^2 - x - 6 = 0
\]

\[
(x-3)(x+2)=0
\]

\[
x - 3 = 0 \text{ or } x + 2 = 0
\]

\[
x = 3 \text{ or } x = -2
\]

For the quadratic inequality, \( x^2 - x - 6 > 0 \) we found zeros 3 and –2 by solving the equation \( x^2 - x - 6 = 0 \). Put these values on a number line and we can see three intervals that we will test in the inequality. We will test one value from each interval.
<table>
<thead>
<tr>
<th>Interval</th>
<th>Test Point</th>
<th>Evaluate in the inequality</th>
<th>True/False</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty,-2)$</td>
<td>$x = -3$</td>
<td>$x^2 - x - 6 &gt; 0$ [ (-3)^2 - (-3) - 6 = 9 + 3 - 6 = 6 &gt; 0 ]</td>
<td>True</td>
</tr>
<tr>
<td>$(-2,3)$</td>
<td>$x = 0$</td>
<td>$x^2 - x - 6 &gt; 0$ [ (0)^2 - (0) - 6 = 0 + 0 - 6 = -6 &lt; 0 ]</td>
<td>False</td>
</tr>
<tr>
<td>$(3,\infty)$</td>
<td>$x = 4$</td>
<td>$x^2 - x - 6 &gt; 0$ [ (4)^2 - (4) - 6 = 16 - 4 - 6 = 6 &gt; 0 ]</td>
<td>True</td>
</tr>
</tbody>
</table>

Thus the intervals $(-\infty,-2)$ or $(3,\infty)$ make up the solution set for the quadratic inequality.

In summary, one way to solve quadratic inequalities is to find the zeros and test a value from each of the intervals surrounding the zeros to determine which intervals make the inequality true.
Example: Solve $2x^2 - 3x + 1 \leq 0$

Solution: First find the zeros by solving the equation,

$$2x^2 - 3x + 1 = 0$$
$$2x^2 - 3x + 1 = 0$$
$$(2x - 1)(x - 1) = 0$$
$$2x - 1 = 0 \text{ or } x - 1 = 0$$

$$x = \frac{1}{2} \text{ or } x = 1$$

Now consider the intervals around the zeros and test a value from each interval in the inequality. The intervals can be seen by putting the zeros on a number line.
<table>
<thead>
<tr>
<th>Interval</th>
<th>Test Point</th>
<th>Evaluate in Inequality</th>
<th>True/False</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, \frac{1}{2}))</td>
<td>(x = 0)</td>
<td>(2x^2 - 3x + 1 &lt; 0)</td>
<td>False</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2(0)^2 - 3(0) + 1 = 0 - 0 + 1 = 1 &gt; 0)</td>
<td></td>
</tr>
<tr>
<td>(\left(\frac{1}{2}, 1\right))</td>
<td>(x = \frac{3}{4})</td>
<td>(2x^2 - 3x + 1 &lt; 0)</td>
<td>True</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) + 1 = \frac{9}{8} - \frac{9}{4} + 1 = \frac{-1}{8} &lt; 0)</td>
<td></td>
</tr>
<tr>
<td>((1, \infty))</td>
<td>(x = 2)</td>
<td>(2x^2 - 3x + 1 &lt; 0)</td>
<td>False</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2(2)^2 - 3(2) + 1 = 8 - 6 + 1 = 3 &gt; 0)</td>
<td></td>
</tr>
</tbody>
</table>

Thus the interval \(\left(\frac{1}{2}, 1\right)\) makes up the solution set for the inequality \(2x^2 - 3x + 1 \leq 0\).
Ex: Solve \( x^2 - 2x < -8 \)

Solution :

\[
x^2 - 2x < -8 \Rightarrow x^2 - 2x + 8 < 0
\]

\[
x^2 - 2x + 8 = 0 \Rightarrow \Delta = 4 - 32 = -28
\]

There are no real roots, thus

Solution set is empty set, \( \{\} \)
Ex: Solve \[(2 - x) \ (x^2 + 4) \ (x^2 - 3x) < 0\]

Solution:

\[2 - x = 0 \implies x = 2\]
\[x^2 + 4 = 0 \implies \text{No real roots}\]
\[x^2 - 3x = 0 \implies x = 0, \ x = 3\]

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>2</th>
<th>3</th>
<th>+∞</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-x</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>x^2+4</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>x^2-3x</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>f(x)</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>
## Practice Problems

<table>
<thead>
<tr>
<th>Expression 1</th>
<th>Expression 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2 + 5x - 24 \leq 0$</td>
<td>$5x^2 - 13x + 6 &lt; 0$</td>
</tr>
<tr>
<td>$12 - x - x^2 &gt; 0$</td>
<td>$9 - x^2 \leq 0$</td>
</tr>
<tr>
<td>$3x^2 + 5x + 2 &lt; 0$</td>
<td>$2x^2 - 5x + 1 &lt; 0$</td>
</tr>
<tr>
<td>$16x^2 - 1 \geq 0$</td>
<td>$x^2 + 5x &lt; -4$</td>
</tr>
<tr>
<td>$3x^2 + 2x + 1 &gt; 0$</td>
<td>$x^2 \leq 2x - 4$</td>
</tr>
</tbody>
</table>
RATIONAL INEQUALITIES

Example: \( \frac{x^2 - x - 6}{x^2 + 4x - 5} \geq 0 \)

\[
\begin{align*}
x^2 - x - 6 &= 0 \\
(x - 3)(x + 2) &= 0 \\
x &= 3, x = -2
\end{align*}
\]

\[
\begin{align*}
x^2 + 4x - 5 &= 0 \\
(x + 5)(x - 1) &= 0 \\
x &= -5, x = 1
\end{align*}
\]

\[
\begin{align*}
x = 3, x = -2, x = -5, x = 1
\end{align*}
\]

These are roots of the expressions.
Use Sign Table,

<table>
<thead>
<tr>
<th>x</th>
<th>-5</th>
<th>-2</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>x+5</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>x+2</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>x−1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>x−3</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>f(x)</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

\[
f(x) = \frac{x^2 - x - 6}{x^2 + 4x - 5} \geq 0
\]

\[
S = (-\infty, -5) \cup [-2, 1) \cup [3, \infty)
\]
Example: \( \frac{x^2 - 4}{x} < 0 \)

\[ x^2 - 4 = (x - 2)(x + 2) = 0, \quad x = -2, x = 2 \]

\( x = 0, x = -2, x = 2 \) are the roots of these expressions.

Use Sign table,

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-2)</th>
<th>0</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x+2 )</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( x )</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( x-2 )</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( f(x) )</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

\[ S = (-\infty, -2) \cup (0, 2) \]
Example: \( \frac{x^2 - 1}{x^2 + 4} < 0 \)

\[ x^2 - 1 = (x - 1)(x + 1) = 0 \]

\[ x = -1, x = 1 \]

\[ x^2 + 4 = 0, \text{ there is no real roots.} \]

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>-1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>x+1</td>
<td></td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>x-1</td>
<td></td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>f(x)</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

\[ S = (-1, 1) \]
Example: \( \frac{x^2 + 1}{x^2 - 4} < 0 \)

\( x^2 + 1 = 0 \) there is no real roots.

\( x^2 - 4 = (x - 2)(x + 2) = 0, \quad x = -2, x = 2 \)

<table>
<thead>
<tr>
<th></th>
<th>-2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>x+2</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>x-2</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>f(x)</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

\[ S = (-2, 2) \]
Example: \( \frac{x^2 + 1}{x^2 + 4} > 0 \quad S = (-\infty, \infty) \)

\[ x^2 + 1 = 0 \quad \text{(always positive) there is no real roots.} \]
\[ x^2 + 4 = 0 \quad \text{(always positive) there is no real roots.} \]

Example: \( \frac{x^2 + 1}{x^2 + 4} < 0 \quad S = \emptyset \)

Exercise: \( \frac{x^2 - 1}{(4 - x^2)(x^2 - 9)} \geq 0 \)
ABSOLUTE VALUE INEQUALITY

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>x</td>
</tr>
<tr>
<td>$</td>
<td>x</td>
</tr>
<tr>
<td>$</td>
<td>x</td>
</tr>
<tr>
<td>$</td>
<td>x</td>
</tr>
</tbody>
</table>
Ex: Solve the following inequalities.

a) \[ |x - 2| < 4 \]
\[ -4 < x - 2 < 4 \]
\[ -2 < x < 6 \]

\[ S = (-2, 6) \]

b) \[ |3 - 2x| \leq 5 \]
\[ -5 \leq 3 - 2x \leq 5 \]
\[ -8 \leq -2x \leq 2 \]
\[ 4 \geq x \geq -1 \]

\[ S = [-1, 4] \]
c) $|x + 5| \geq 7$

$x + 5 \leq -7$ or $x + 5 \geq 7$

$x \leq -12$ or $x \geq 2$

\[ S = (-\infty, -12] \cup [2, \infty) \]

d) $|3x - 1| < 5$

$-5 < 3x - 1 < 5$

$-4 < 3x < 6$

$-\frac{4}{3} < x < 2$

\[ S = \left(-\frac{4}{3}, 2\right) \]
e) \(|2x - 5| \geq 3\)

\[
2x - 5 \geq 3 \quad \text{or} \quad 2x - 5 \leq -3
\]

\[
x \geq 4 \quad \text{or} \quad x \leq 1
\]

\[
S = (-\infty, 1] \cup [4, \infty)
\]

f) \(|4x - 3| \leq -2\)

\[
S = \emptyset
\]