0.5 FACTORING

If two or more expressions are multiplied together, the expressions are called factors of the product. Thus, if \( c = ab \), then \( a \) and \( b \) are both factors of the product \( c \).

**Rules for Factoring**

1. \( xy + xz = x(y + z) \)  
   (common factor)
2. \( x^2 + (a + b)x + ab = (x + a)(x + b) \)
3. \( abx^2 + (ad + cb)x + cd = (ax + c)(bx + d) \)
4. \( x^2 + 2ax + a^2 = (x + a)^2 \)  
   (perfect-square trinomial)
5. \( x^2 - 2ax + a^2 = (x - a)^2 \)  
   (perfect-square trinomial)
6. \( x^2 - a^2 = (x + a)(x - a) \)  
   (difference of two squares)
7. \( x^3 + a^3 = (x + a)(x^2 - ax + a^2) \)  
   (sum of two cubes)
8. \( x^3 - a^3 = (x - a)(x^2 + ax + a^2) \)  
   (difference of two cubes)
EXAMPLE 1 Common Factors

a. **Factor** $3k^2x^2 + 9k^3x$ **completely.**

**Solution:** Since $3k^2x^2 = (3k^2x)(x)$ and $9k^3x = (3k^2x)(3k)$, each term of the original expression contains the common factor $3k^2x$. Thus, by Rule 1,

$$3k^2x^2 + 9k^3x = 3k^2x(x + 3k)$$

b. **Factor** $8a^5x^2y^3 - 6a^2b^3yz - 2a^4b^4xy^2z^2$ **completely.**

**Solution:**

$$8a^5x^2y^3 - 6a^2b^3yz - 2a^4b^4xy^2z^2 = 2a^2y(4a^3x^2y^2 - 3b^3z - a^2b^4xyz^2)$$
EXAMPLE 2 \hspace{1cm} \textbf{Factoring Trinomials}

a. \textit{Factor} $3x^2 + 6x + 3$ \textit{completely}.

\textit{Solution:} First we remove a common factor. Then we factor the resulting expression completely. Thus, we have

\[ 3x^2 + 6x + 3 = 3(x^2 + 2x + 1) \]
\[ = 3(x + 1)^2 \quad \text{(Rule 4)} \]

b. \textit{Factor} $x^2 - x - 6$ \textit{completely}.

\textit{Solution:} Since $(x + a)(x + b) = x^2 + (a + b)x + ab$, it follows that

\[ x^2 + (-1)x + (-6) = x^2 + (a + b)x + ab \]

By equating corresponding coefficients, we want \[ a + b = -1 \quad \text{and} \quad ab = -6 \]

If $a = -3$ and $b = 2$, then both conditions are met, and hence,

\[ x^2 - x - 6 = (x - 3)(x + 2) \]
c. Factor $x^2 - 7x + 12$ completely.

   Solution:

   $x^2 - 7x + 12 = (x - 3)(x - 4)$

d. $x^2 - 6x + 9 = (x - 3)^2$

e. $x^4 - 1 = (x^2 + 1)(x^2 - 1)$
   
   $= (x^2 + 1)(x + 1)(x - 1)$

f. $8 - x^3 = (2)^3 - (x)^3 = (2 - x)(4 + 2x + x^2)$
0.6 FRACTIONS

EXAMPLE 1  Simplifying Fractions

a. Simplify \( \frac{x^2 - x - 6}{x^2 - 7x + 12} \).

Solution: First, we completely factor the numerator and the denominator:

\[
\frac{x^2 - x - 6}{x^2 - 7x + 12} = \frac{(x - 3)(x + 2)}{(x - 3)(x - 4)}
\]

Dividing both numerator and denominator by the common factor \(x - 3\), we have

\[
\frac{x^2 - x - 6}{x^2 - 7x + 12} \times \frac{1}{1} = \frac{(x - 3)(x + 2)}{(x - 3)(x - 4)} \times \frac{1}{1} = \frac{x + 2}{x - 4}
\]

The process of eliminating the common factor \(x - 3\) is commonly referred to as “cancellation.”
b. Simplify \( \frac{2x^2 + 6x - 8}{8 - 4x - 4x^2} \).

Solution:

\[
\frac{2x^2 + 6x - 8}{8 - 4x - 4x^2} = \frac{2(x^2 + 3x - 4)}{4(2 - x - x^2)} = \frac{2(x - 1)(x + 4)}{4(1 - x)(2 + x)}
\]

\[
= \frac{2(x - 1)(x + 4)}{2(2)[(-1)(x - 1)](2 + x)}
\]

\[
= \frac{x + 4}{-2(2 + x)} = -\frac{x + 4}{2(x + 2)}
\]
Multiplication and Division of Fractions

The rule for multiplying $\frac{a}{b}$ by $\frac{c}{d}$ is

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

**EXAMPLE 2**  
Multiplying Fractions

$$\frac{x^2 - 4x + 4}{x^2 + 2x - 3} \cdot \frac{6x^2 - 6}{x^2 + 2x - 8} = \frac{[(x - 2)^2][6(x + 1)(x - 1)]}{[(x + 3)(x - 1)][(x + 4)(x - 2)]}$$

$$= \frac{6(x - 2)(x + 1)}{(x + 3)(x + 4)}$$
EXAMPLE 3  Dividing Fractions

a. \[ \frac{x}{x + 2} \div \frac{x + 3}{x - 5} = \frac{x}{x + 2} \cdot \frac{x - 5}{x + 3} = \frac{x(x - 5)}{(x + 2)(x + 3)} \]

b. \[ \frac{x - 5}{2x} = \frac{x - 5}{x - 3} \cdot \frac{1}{2x} = \frac{x - 5}{2x(x - 3)} \]

c. \[ \frac{4x}{x^2 - 1} \cdot \frac{x - 1}{2x^2 + 8x} = \frac{4x(x - 1)}{[(x + 1)(x - 1)][2x(x + 4)]} = \frac{2}{(x + 1)(x + 4)} \]
Rationalizing the Denominator

Sometimes the denominator of a fraction has two terms and involves square roots, such as $2 - \sqrt{3}$ or $\sqrt{5} + \sqrt{2}$. The denominator may then be rationalized by multiplying by an expression that makes the denominator a difference of two squares. For example,

\[
\frac{4}{\sqrt{5} + \sqrt{2}} = \frac{4}{\sqrt{5} + \sqrt{2}} \cdot \frac{\sqrt{5} - \sqrt{2}}{\sqrt{5} - \sqrt{2}}
\]

\[
= \frac{4(\sqrt{5} - \sqrt{2})}{(\sqrt{5})^2 - (\sqrt{2})^2} = \frac{4(\sqrt{5} - \sqrt{2})}{5 - 2}
\]

\[
= \frac{4(\sqrt{5} - \sqrt{2})}{3}
\]
EXAMPLE 4  Rationalizing Denominators

a. \[ \frac{x}{\sqrt{2} - 6} = \frac{x}{\sqrt{2} - 6} \cdot \frac{\sqrt{2} + 6}{\sqrt{2} + 6} = \frac{x(\sqrt{2} + 6)}{(\sqrt{2})^2 - 6^2} \]

\[ = \frac{x(\sqrt{2} + 6)}{2 - 36} = \frac{x(\sqrt{2} + 6)}{-34} \]

b. \[ \frac{\sqrt{5} - \sqrt{2}}{\sqrt{5} + \sqrt{2}} = \frac{\sqrt{5} - \sqrt{2}}{\sqrt{5} + \sqrt{2}} \cdot \frac{\sqrt{5} - \sqrt{2}}{\sqrt{5} - \sqrt{2}} = \frac{(\sqrt{5} - \sqrt{2})^2}{5 - 2} = \frac{5 - 2\sqrt{5}\sqrt{2} + 2}{3} = \frac{7 - 2\sqrt{10}}{3} \]
**Addition and Subtraction of Fractions**

If we add or subtract two fractions having a common denominator then the resulting is a fraction whose denominator is the common denominator

\[
\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}
\]

**EXAMPLE 5**

**Adding and Subtracting Fractions**

a. \[
\frac{p^2 - 5}{p - 2} + \frac{3p + 2}{p - 2} = \frac{(p^2 - 5) + (3p + 2)}{p - 2} = \frac{p^2 + 3p - 3}{p - 2}
\]
b. \[
\frac{x^2 - 5x + 4}{x^2 + 2x - 3} - \frac{x^2 + 2x}{x^2 + 5x + 6} = \frac{(x - 1)(x - 4)}{(x - 1)(x + 3)} - \frac{x(x + 2)}{(x + 2)(x + 3)}
\]
\[
= \frac{x - 4}{x + 3} - \frac{x}{x + 3}
\]
\[
= \frac{(x - 4) - x}{x + 3} = -\frac{4}{x + 3}
\]

c. \[
\frac{x^2 + x - 5}{x - 7} - \frac{x^2 - 2}{x - 7} + \frac{-4x + 8}{x^2 - 9x + 14}
\]
\[
= \frac{x^2 + x - 5}{x - 7} - \frac{x^2 - 2}{x - 7} + \frac{-4(x - 2)}{(x - 7)(x - 2)}
\]
\[
= \frac{(x^2 + x - 5) - (x^2 - 2) + (-4)}{x - 7}
\]
\[
= \frac{x - 7}{x - 7} = 1
\]
To add (or subtract) two fractions with different denominators, use the fundamental principle of fractions to rewrite the fractions as equivalent fractions that have the same denominator. Then proceed with the addition (or subtraction) by the method just described.

\[
\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}
\]

For example, \(\frac{2}{x^3(x - 3)} + \frac{3}{x(x - 3)^2}\)

we can convert the first fraction into an equivalent fraction by multiplying the numerator and denominator by \(x - 3\):

\[
\frac{2(x - 3)}{x^3(x - 3)^2}
\]
we can convert the second fraction by multiplying the numerator and denominator by \(x^2\):

\[
\frac{3x^2}{x^3(x - 3)^2}
\]

These fractions have the same denominator. Hence,

\[
\frac{2}{x^3(x - 3)} + \frac{3}{x(x - 3)^2} = \frac{2(x - 3)}{x^3(x - 3)^2} + \frac{3x^2}{x^3(x - 3)^2}
\]

\[
= \frac{3x^2 + 2x - 6}{x^3(x - 3)^2}
\]
EXAMPLE 6    Adding and Subtracting Fractions

a. Subtract: \( \frac{t}{3t + 2} - \frac{4}{t - 1} = \frac{t(t - 1)}{(3t + 2)(t - 1)} - \frac{4(3t + 2)}{(3t + 2)(t - 1)} \)

\[ = \frac{t(t - 1) - 4(3t + 2)}{(3t + 2)(t - 1)} \]

\[ = \frac{t^2 - t - 12t - 8}{(3t + 2)(t - 1)} = \frac{t^2 - 13t - 8}{(3t + 2)(t - 1)} \]

b. Add: \( \frac{4}{q - 1} + 3 = \frac{4}{q - 1} + \frac{3(q - 1)}{q - 1} \)

\[ = \frac{4 + 3(q - 1)}{q - 1} = \frac{3q + 1}{q - 1} \]
EXAMPLE 7  Subtracting Fractions

\[
\frac{x - 2}{x^2 + 6x + 9} - \frac{x + 2}{2(x^2 - 9)} = \frac{x - 2}{(x + 3)^2} - \frac{x + 2}{2(x + 3)(x - 3)}
\]

\[\text{LCD} = 2(x + 3)^2(x - 3)\]

\[
= \frac{(x - 2)(2)(x - 3)}{(x + 3)^2(2)(x - 3)} - \frac{(x + 2)(x + 3)}{2(x + 3)(x - 3)(x + 3)}
\]

\[
= \frac{(x - 2)(2)(x - 3) - (x + 2)(x + 3)}{2(x + 3)^2(x - 3)}
\]

\[
= \frac{2(x^2 - 5x + 6) - (x^2 + 5x + 6)}{2(x + 3)^2(x - 3)}
\]

\[
= \frac{2x^2 - 10x + 12 - x^2 - 5x - 6}{2(x + 3)^2(x - 3)}
\]

\[
= \frac{x^2 - 15x + 6}{2(x + 3)^2(x - 3)}
\]