1. Find the inverse of the partitioned \( m \times m \) matrix
\[
\begin{bmatrix}
B & D \\
0 & C
\end{bmatrix}
\]
where \( B \) is a nonsingular \( r \times r \) matrix, \( C \) is a nonsingular \( (m-r) \times (m-r) \) matrix, \( D \) is an \( r \times (m-r) \) matrix, and \( 0 \) is the \( (m-r) \times r \) matrix of zeros.

2. Two matrices \( A \) and \( B \) commute if \( AB = BA \). Show that the only \( m \times m \) matrices \( A \) that commute with every \( m \times m \) matrix \( B \) are those of the form \( \alpha I_m \) for some scalar \( \alpha \). (Hint: Consider the matrix \( B = E^{ij} \) having all zeroes except for a 1 in the \((i, j)^{th}\) position.)

3. Murty, Problem 3.17. (You need to assume that the system \( Ax = b \) has at least one solution.)
1. Use the Phase I/Phase II Simplex Method to solve Murty Problem 2.6, 2.11, 2.15, and 2.18. State which type of LP each of these is. Further, if an optimal solution is reached, give the optimal solution, and if an unbounded LP is discovered, give the set of solutions having no lower bound.

2. Consider the following LP

\[
\begin{align*}
\text{min } z &= -2x_1 - 3x_2 - 5x_3 - 4x_4 \\
&\quad x_1 + 2x_2 + 3x_3 + x_4 \leq 5 \\
&\quad x_1 + x_2 + 2x_3 + 3x_4 \leq 3 \\
&\quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0
\end{align*}
\]

(P)

(a) Solve (P) using the Simplex Method.
(b) Give the objective function in the equivalent form indicated by this tableau. Use this equation to derive a simple system that describes all optimal solutions to (P)
(c) Pivot to every basic optimal tableau for (P). List the set of all basic feasible optimal solutions.

3. Cycling and Bland’s Rule: Consider the following LP

\[
\begin{align*}
\text{min } z &= -6x_1 - 5x_2 + 30x_3 + 2x_4 + 3x_5 \\
&\quad -3x_1 + 2x_2 - 21x_3 - 5x_4 + 24x_5 \leq 0 \\
&\quad 3x_1 + x_2 - 6x_3 - x_4 + 3x_5 \leq 0 \\
&\quad -3x_1 + 2x_2 - 12x_3 + x_4 - 12x_5 \leq 1 \\
&\quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0, \quad x_5 \geq 0
\end{align*}
\]

Apply the simplex method three times to this problem, the first time using tie-breaking rule R1 for all pivots, the second time using tie-breaking rule R2 for all pivots, and the third time using tie-breaking rule R3 for all pivots.

R1: For the entering variable, choose the most negative objective-function-row value. For the leaving variable, choose the topmost row among those having the minimum ratio.

R2: For the entering variable, choose the leftmost column among those having negative objective-function-row value. For the leaving variable, choose the topmost row among those having the minimum ratio.

R3: Bland’s Rule.

Show that using first two rules the simplex method never reaches an optimal solution, whereas using Bland’s rule it does.

Extra Credit Problem: Murty, Problem 2.37
1. Find the Farkas’ Lemma “dual” solution for (infeasible) Murty 2.13 by applying the Row Progressive Phase I Simplex Method with an added identity matrix.

2. Re-solve Murty 2.11 and 2.18 using the Artificial Phase I/Phase II Simplex Method.

3. Solve the following problem, using the Phase II Revised Simplex Method. Show the equations you used at each step to get the appropriate tableau values (matrix equations are fine).

\[
\begin{align*}
\text{max } z &= 5x_1 + 8x_2 + 7x_3 + 4x_4 + 6x_5 \\
2x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 &\leq 20 \\
3x_1 + 5x_2 + 4x_3 + 2x_4 + 4x_5 &\leq 30 \\
x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0, \quad x_5 \geq 0
\end{align*}
\]

4. Solve Murty 5.3 using the Phase I/Phase II Revised Simplex Method. (Ignore the instruction to start with a given initial basis.) Again, show your equations at each step.
5. **Farkas’ Lemma for Linear Inequality Systems in Nonnegative Variables**: Either the system:

\[ Ax \leq b \quad x \geq 0 \]

has a solution or the system

\[ yA \geq 0, \quad y \geq 0, \quad yb < 0 \]

has a solution, but not both.

(a) Prove this by reducing it to the standard Farkas’ Lemma.

(b) Demonstrate it with Murty 2.28, again applying the Row Progressive Phase I Simplex Method.

6. Sometimes the Artificial Phase I Simplex Method ends in an optimal tableau for which \( w^* = 0 \) but not all of the artificial variables are nonbasic. (For example, if we add the constraint \(-2x_1 = 0\) to the system in Murty Problem 2.18 and resolve using the Artificial Phase I Method, the artificial variable for this constraint remains basic.) Assuming there are no redundant constraints in the original equality system, tell how to obtain, from any basic feasible artificial tableau having \( w = 0 \), a basic feasible tableau for the original problem (that is, no artificial variables are present). What happens when redundant constraints are present?