Linear Programming Models
Woody’s Problem:
A Resource Allocation Model

Woody’s Furniture Company makes chairs and tables. Chairs are made entirely out of pine, and require 8 linear feet of pine per chair. Tables are made of pine and mahogany, with tables requiring 12 linear feet of pine and 15 linear feet of mahogany. Chairs require 3 hours of labor to produce and tables require 6 hours of labor. Chairs provide $35 profit and tables provide $60 profit.

Woody has 120 linear feet of pine and 60 linear feet of mahogany delivered each day, and has a work force of 6 carpenters each of whom put in an 8-hour day. How should Woody use his resources to provide the largest daily profits?

The mathematical description:

\[
x_1 = \text{chairs per day, } x_2 = \text{tables per day},
\]

\[
\max z = 35x_1 + 60x_2 \quad \text{profits}
\]

\[
8x_1 + 12x_2 \leq 120 \quad \text{pine}
\]
\[
15x_2 \leq 60 \quad \text{mahogany}
\]
\[
3x_1 + 6x_2 \leq 48 \quad \text{labor}
\]

\[
x_1 \geq 0 \quad x_2 \geq 0 \quad \text{nonnegativity constraints}
\]
Linear Programming Models

Each of the models above are examples of linear programs. Linear programs are characterized by the following properties:

**optimization problem/mathematical program:** The problem involves finding the best value for a given objective, subject to a given set of constraints.

**deterministic:** The behavior of the model is completely determined, that is, there is no probability contributing to the objective or constraints.

**proportionality:** Each variable contributes in direct proportion to its value.

**additivity:** The variables in the objective and each constraint contribute the sum of the contributions of each variable.

**divisibility:** The variables can take on continuous values subject to the constraints.

Any function satisfying the proportionality and additivity properties is called a linear function, and will always have the form

\[ f(x_1, \ldots, x_n) = a_1x_1 + \ldots + a_nx_n( + d). \]
The Mathematical Description

\[
\begin{align*}
\max \quad & z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\
\min \quad & \begin{cases}
  a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n 
  \leq b_i & i = 1, \ldots, m \\
  \geq b_i & i = 1, \ldots, m \\
  \geq 0 & j = 1, \ldots, n \\
  (\leq 0) & j = 1, \ldots, n
\end{cases}
\end{align*}
\]

\(x_j\) are the variables of the problem, and are allowed to take on any set of real values that satisfy the constraints.

\(c_j, b_i,\) and \(a_{ij}\) are parameters of the problem, and (along with dimensions \(n, m,\) and objective/row/variable types) provide the precise description of a particular instance of the LP model that you wish to solve.

\(c_j\) are the objective/cost/profit coefficients.

\(b_i\) are the right-hand-side/resource/demand coefficients.

\(a_{ij}\) are the proportionality/production/activity coefficients or coefficients of variation.
Goals of the Course

• To study how linear programs are solved, and the properties of LPs and optimal solutions to LPs
• To present the most effective computational methods for solving LPs
• To study the geometry of linear systems, and see how it relates to the computational methods for solving LPs
• To understand the effect changes in the parameters of an LP have on its optimal solutions
• To see the powerful role duality has in understanding, solving, and applying LPs in many contexts
• To see how LPs — and the underlying theory — are used to solve more complex types of optimization problems such as multiobjective linear programs, integer linear programs and nonlinear programs
Two Special Forms of LPs

The standard (equality form) min LP:

\[
\begin{align*}
    \text{min } z &= c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\
    a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= b_1 \\
    a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= b_2 \\
    &\vdots \\
    a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &= b_m \\
    x_1 &\geq 0 \quad x_2 &\geq 0 \quad \ldots \quad x_n &\geq 0.
\end{align*}
\]

An inequality form max LP:

\[
\begin{align*}
    \text{max } z &= c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\
    a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &\leq b_1 \\
    a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &\leq b_2 \\
    &\vdots \\
    a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &\leq b_m
\end{align*}
\]
Definitions and Issues

**feasible solution:** A specific set of values of the variables that satisfy the constraints of that instance.

**feasible region:** The set of all feasible solutions for that instance.

**optimal solution:** Any feasible solution whose objective function value is the greatest (for a max problem) or smallest (for a min problem) of all points in the feasible region.

**Some issues:**

- Does a particular instance of an LP have *any* feasible point?
- Does the instance have an *optimal* solution?
- How can we describe the feasible region? What properties does it have?
- How will we know that a solution to a particular instance is in fact an optimal solution for that instance?
- Can we give an *algorithm* to find feasible/optimal solutions for any instance of an LP, or to determine that none exist for that instance?
- How *efficient* is that algorithm, particularly for solving large LP instances?
A Graphical View of LPs

We can gain insight into the dynamics of LPs by looking at 2-variable problems. Consider Woody’s problem written in inequality max form:

\[
\begin{align*}
\text{max} \; z &= 35x_1 + 60x_2 \\
8x_1 &+ 12x_2 \leq 120 \\
15x_2 &\leq 60 \\
3x_1 &+ 6x_2 \leq 48 \\
-x_1 &\leq 0 \\
-x_2 &\leq 0
\end{align*}
\]

Draw a picture of this LP, by performing the following steps:

1. Take each of the constraints (including the nonnegativity constraints), and draw the line describing the associated equality.

2. Determine which side of each equality line satisfies the inequality constraint. The feasible region is set of points which lie on the correct side of all of the equality lines.

3. For particular objective function value \( z_0 \), the isocost line at value \( z_0 \) is described by the equality line \( c_1 x_1 + c_2 x_2 = z_0 \).

4. Slide this isocost line parallel and in the direction of improving objective value until the last point at which it still intersects the feasible region.

5. Any feasible point in this final intersection is an optimal solution.
Example

\[
\begin{align*}
\text{max } z & = 35x_1 + 60x_2 \\
(C1) \quad 8x_1 + 12x_2 & \leq 120 \\
(C2) \quad 15x_2 & \leq 60 \\
(C3) \quad 3x_1 + 6x_2 & \leq 48 \\
-x_1 & \leq 0 \\
-x_2 & \leq 0
\end{align*}
\]
Special subsets of the feasible region

**interior**: set of points having neighborhoods that lie entirely in the feasible region. Equivalently, an interior point is one that satisfies all inequalities *strictly*.

**boundary**: set of feasible points not in the interior, that is, which satisfy one or more inequalities at equality

**extreme point/corner point/vertex**: boundary point lying on \( n \) equalities, where \( n \) = the dimension of the problem.

LP types

**optimal solution**: one can find a feasible solution for which no other solution has a better objective function value.

**infeasible LP**: the constraints of the problem are such that the feasible region is empty, in other words, there is no feasible solution.

**unbounded LP**: (maximization problem) the objective and constraints of the problem are such that there is no upper bound on the objective function values of the feasible points. That is, for every feasible solution there exists another feasible solution with larger objective function value. (A symmetric definition applies for minimization problems.)
Three Types of Linear Programs

Optimal Solution

\[
\text{max } z = 35x_1 + 60x_2 \\
(C1) \quad 8x_1 + 12x_2 \leq 120 \\
(C2) \quad 15x_2 \leq 60 \\
(C3) \quad 3x_1 + 6x_2 \leq 48 \\
\quad -x_1 \leq 0 \\
\quad -x_2 \leq 0
\]
Infeasible LP

\[ \text{max } z = 35x_1 + 60x_2 \]

\[(C1) \quad 8x_1 + 12x_2 \geq 120 \]

\[(C2) \quad 15x_2 \geq 60 \]

\[(C3) \quad 3x_1 + 6x_2 \leq 48 \]

\[-x_1 \leq 0 \]

\[-x_2 \leq 0 \]
Unbounded LP

\[
\begin{align*}
\text{max } z &= 35x_1 + 60x_2 \\
(C1) &\quad -8x_1 + 12x_2 \leq 120 \\
(C2) &\quad -20x_1 + 15x_2 \leq 60 \\
(C3) &\quad 3x_1 - 6x_2 \leq 48 \\
&\quad -x_1 \leq 0 \\
&\quad -x_2 \leq 0
\end{align*}
\]
The Graphical Method in 3-Dimensions

Consider the Woody’s Problem with a third item, say desks, represented by variable $x_3$, requiring 16 l.f. of pine, 20 l.f. of mahogany, 9 hrs. of labor, and giving a profit of $75:

$$\begin{align*}
\text{max } z &= 35x_1 + 60x_2 + 75x_3 \\
8x_1 &+ 12x_2 + 16x_3 \leq 120 \\
15x_2 &+ 20x_3 \leq 60 \\
3x_1 &+ 6x_2 + 9x_3 \leq 48 \\
x_1 &\geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0
\end{align*}$$

This also has a graphical representation, now in 3-dimensional Euclidean space:
The level set is now a flat sheet perpendicular to the objective function vector \((35,60,75)\). Adding this to the picture, we get

We can slide this level set to the optimal solution \((12,2,0)\), although this is difficult to conceptualize even in 3 dimensions, and becomes completely unwieldy in higher dimensions.
Transforming a General LP into Equality Form

**maximization problems:** Replace

\[ \max z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \]

with

\[ \min z^- = - c_1 x_1 - c_2 x_2 - \ldots - c_n x_n \]

**\leq constraints:** Add a nonnegative slack variable indicating the difference between the LHS value and the \( b_i \) value. Specifically, replace

\[ a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n \leq b_i \]

with

\[ a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n + x_{n+i} = b_i, \quad x_{n+i} \geq 0 \]

**\geq constraints:** Subtract the slack variable in that row. Specifically, replace

\[ a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n \geq b_i \]

with

\[ a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n - x_{n+i} = b_i, \quad x_{n+i} \geq 0 \]

**unrestricted variables:** Replace unrestricted \( x_i \) by

\[ x_i = x_i^+ - x_i^-, \quad x_i^+ \geq 0, \quad x_i^- \geq 0 \]

**negative variables:** Replace \( x_i \leq 0 \) by

\[ x_i^- = -x_i, \quad x_i \geq 0. \]
Examples

Woody’s problem can be put into standard min form by simply negating the objective function and adding slack variables to each of the inequalities

\[
\begin{align*}
\text{min } z^- &= -35x_1 - 60x_2 \\
8x_1 + 12x_2 + x_3 &= 120 \\
15x_2 + x_4 &= 60 \\
3x_1 + 6x_2 + x_5 &= 48 \\
x_1 &\geq 0 \\
x_2 &\geq 0 \\
x_3 &\geq 0 \\
x_4 &\geq 0 \\
x_5 &\geq 0
\end{align*}
\]

Now consider the following modification of Woody’s problem:

\[
\begin{align*}
\text{max } z &= 35x_1 + 60x_2 \\
-8x_1 + 12x_2 &= 120 \\
-20x_1 + 15x_2 &\geq 60 \\
2x_1 - 6x_2 &\leq 48 \\
x_1 \text{ unrest. } x_2 &\leq 0
\end{align*}
\]

Here we *subtract* a slack in Row 2, *add* a slack in Row 3, negate the objective function, replace \(x_1\) with \(x_1^+ - x_1^-\) and replace \(x_2\) by \(-x_2^-\) to get standard max LP

\[
\begin{align*}
\text{min } z &= -35x_1^+ + 35x_1^- + 60x_2^- \\
-8x_1^+ + 8x_1^- - 12x_2^- &= 120 \\
-20x_1^+ + 20x_1^- - 15x_2^- - x_4 &= 60 \\
2x_1^+ - 2x_1^- + 6x_2^- + x_5 &= 48 \\
x_1^+ &\geq 0 \\
x_1^- &\geq 0 \\
x_2^- &\geq 0 \\
x_4 &\geq 0 \\
x_5 &\geq 0
\end{align*}
\]
Basic Linear Algebra
Scalars and Vectors

**Scalars:** A *scalar* is any real number.

**Vectors:**

A *row m-vector* is written

\[ y = [y_1, y_2, \ldots, y_m] \]

A *column n-vector* is written

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}
\]

Vectors are indexed \( x^k, y^k \), etc.

**Operations on vectors:**

The *transpose* of a column \( n \)-vector (row \( n \)-vector) \( c \) is the row \( n \)-vector (column \( n \)-vector) \( c^T \) having the same component values as \( c \).

The *sum* \( c + d \) of two column \( n \)-vectors (two row \( n \)-vectors) is the column \( n \)-vector (row \( n \)-vector) whose components are the sums of the corresponding components of \( c \) and \( d \).

The *(dot) product* of a row \( n \)-vector \( c \) and a column \( n \)-vector \( x \) is

\[ cx = \langle c, x \rangle = \sum_{i=1}^n c_i x_i \]
Matrices

An $m \times n$ matrix $A = [a_{ij}]$ is written as

$$A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}$$

The $i^{th}$ row of $A$ is denoted $A_i$ and the $j^{th}$ column of $A$ is denoted $A_j$.

Operations on matrices: Let $A$ be an $m \times n$ matrix, $\alpha$ a scalar, $y$ a row $m$-vector, and $x$ a column $n$-vector. Then we have

$$\alpha A = A\alpha = [\alpha a_{ij}]$$

$$Ax = \begin{bmatrix}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
    \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{bmatrix}$$

$$yA = \begin{bmatrix}
    y_1a_{11} + y_2a_{21} + \cdots + y_ma_{m1}, \\
    y_1a_{12} + y_2a_{22} + \cdots + y_ma_{m2}, \\
    \ldots, y_1a_{1n} + y_2a_{2n} + \cdots + y_ma_{mn}
\end{bmatrix}$$

Given two $m \times n$ matrices $A$ and $B$, the sum of $A$ and $B$ is the $m \times n$ matrix $[a_{ij} + b_{ij}]$.

Given $m \times r$ matrix $A$ and $r \times n$ matrix $B$, the product $AB$ is the $m \times n$ matrix $D = [d_{ij}]$ with

$$d_{ij} = A_i.B_j = \sum_{k=1}^{r} a_{ik}b_{kj}$$
Examples

\[
\begin{bmatrix}
0 & 60 & 35
\end{bmatrix}
\begin{bmatrix}
30 \\
2 \\
12
\end{bmatrix} = 540
\]

\[
\begin{bmatrix}
15/4 & 1 & -10 \\
-1/4 & 0 & 2/3 \\
1/2 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
120 \\
60 \\
48
\end{bmatrix} = \begin{bmatrix}
30 \\
2 \\
12
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 60 & 35
\end{bmatrix}
\begin{bmatrix}
15/4 & 1 & -10 \\
-1/4 & 0 & 2/3 \\
1/2 & 0 & -1
\end{bmatrix} = \begin{bmatrix}
5/2 & 0 & 5
\end{bmatrix}
\]

\[
\begin{bmatrix}
15/4 & 1 & -10 \\
-1/4 & 0 & 2/3 \\
1/2 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
8 & 12 & 1 & 0 & 0 \\
0 & 15 & 0 & 1 & 0 \\
3 & 6 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 15/4 & 1 & -10 \\
0 & 1 & -1/4 & 0 & 2/3 \\
1 & 0 & 1/2 & 0 & -1
\end{bmatrix}
\]
The transpose of $A$ is the $n \times m$ matrix

$$A^T = \begin{bmatrix}
  a_{11} & a_{21} & \cdots & a_{m1} \\
  a_{12} & a_{22} & \cdots & a_{m2} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{1n} & a_{2n} & \cdots & a_{mn}
\end{bmatrix}$$

The $m \times m$ identity matrix is the matrix

$$I = I_m = \begin{bmatrix}
  1 & 0 & \cdots & 0 & 0 \\
  0 & 1 & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 1 & 0 \\
  0 & 0 & \cdots & 0 & 1
\end{bmatrix}$$

Some properties of multiplication and addition (assume all matrices have the appropriate dimension):

1. (Associativity) $A(BC) = (AB)C$
2. (Distributivity) $A(B + C) = AB + AC$, $(B + C)D = BD + CD$
3. (Identity) $AI = A$, $IB = B$
4. (Distributivity of transpose) $(AB)^T = B^T A^T$
5. (Commutativity of scalars) for scalar $\alpha$, $\alpha AB = A\alpha B = AB\alpha$. (Note that this does not hold for vectors or matrices.)

The inverse of an $m \times m$ matrix $B$ is the $m \times m$ matrix $B^{-1}$ such that

$$B^{-1}B = BB^{-1} = I.$$  

An $m \times m$ matrix $B$ having an inverse is called invertible.
We can combine several operations together (as long as the dimensions of the appropriate operations match), for example

\[
\begin{bmatrix}
-35, -60, 0, 0, 0
\end{bmatrix} + \begin{bmatrix}
0, 60, 35
\end{bmatrix} = \begin{bmatrix}
0, 60, 35
\end{bmatrix}
\]

Equivalently, using associativity we can multiply the two matrices together first

\[
\begin{bmatrix}
-35, -60, 0, 0, 0
\end{bmatrix} + \begin{bmatrix}
0, 60, 35
\end{bmatrix} = \begin{bmatrix}
0, 60, 35
\end{bmatrix}
\]
A submatrix of $m \times n$ matrix $A = [a_{ij}]$ is any matrix of the form
\[
\begin{bmatrix}
  a_{i_1j_1} & a_{i_1j_2} & \ldots & a_{i_1j_s} \\
  a_{i_2j_1} & a_{i_2j_2} & \ldots & a_{i_2j_s} \\
  \vdots & \vdots & & \vdots \\
  a_{i_rj_1} & a_{i_rj_2} & \ldots & a_{i_rj_s}
\end{bmatrix}
\]
for sequences of distinct indices $i_1, \ldots, i_r$ and $j_1, \ldots, j_s$ chosen from $1, \ldots, m$ and $1, \ldots, n$, respectively.

A partition of a matrix $A$ is an arrangement
\[
\begin{bmatrix}
  A^{11} & \ldots & A^{1k} \\
  \vdots & & \vdots \\
  A^{l1} & \ldots & A^{lk}
\end{bmatrix}
\]
where the $A^{ij}$ are submatrices of $A$ whose row and column indices partition those of $A$. Partitioned vector and matrix multiplication looks similar to the component version — order, however, being critical — e.g.

\[
\begin{bmatrix}
  A^1 & A^2 \\
\end{bmatrix}
\begin{bmatrix}
  x^1 \\
  x^2
\end{bmatrix} = A^1x^1 + A^2x^2,
\begin{bmatrix}
  A^1 \\
  A^2
\end{bmatrix}x = \begin{bmatrix}
  A^1x \\
  A^2x
\end{bmatrix},
\]

\[
\begin{bmatrix}
  y^1 & y^2 \\
\end{bmatrix}
\begin{bmatrix}
  B^1 \\
  B^2
\end{bmatrix} = y^1B^1 + y^2B^2,
[y^1 y^2]B = [y^1 B \ y^2 B],
\]

\[
\begin{bmatrix}
  A^{11} & A^{12} \\
  A^{21} & A^{22}
\end{bmatrix}
\begin{bmatrix}
  B^{11} & B^{12} \\
  B^{21} & B^{22}
\end{bmatrix} = \begin{bmatrix}
  A^{11}B^{11} + A^{12}B^{21} & A^{11}B^{12} + A^{12}B^{22} \\
  A^{21}B^{11} + A^{22}B^{21} & A^{21}B^{12} + A^{22}B^{22}
\end{bmatrix}
\]
Example

Let

\[
A = \begin{bmatrix}
8 & 12 & 1 & 0 & 0 \\
0 & 15 & 0 & 1 & 0 \\
3 & 6 & 0 & 0 & 1
\end{bmatrix},
\]

\[
x = \begin{bmatrix}
9 \\
3 \\
12 \\
15 \\
3
\end{bmatrix}
\]

and

\[
b = Ax = \begin{bmatrix}
120 \\
60 \\
48
\end{bmatrix}.
\]

Partitioning \(A\) by rows \([3, 1][2]\) and by columns \([4, 2, 1][3, 5]\) gives the four submatrices

\[
A^{11} = \begin{bmatrix}
0 & 6 & 3 \\
0 & 12 & 8
\end{bmatrix},
A^{12} = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

\[
A^{21} = \begin{bmatrix}
1 & 15 & 0
\end{bmatrix},
A^{22} = \begin{bmatrix}
0 & 0
\end{bmatrix}.
\]

The corresponding partition of \(x\) is

\[
x^1 = \begin{bmatrix}
15 \\
3 \\
9
\end{bmatrix},
x^2 = \begin{bmatrix}
12 \\
3
\end{bmatrix}.
\]

If we set

\[
b^1 = A^{11}x^1 + A^{12}x^2 = \begin{bmatrix}
45 \\
108
\end{bmatrix} + \begin{bmatrix}
3 \\
12
\end{bmatrix} = \begin{bmatrix}
48 \\
120
\end{bmatrix}
\]

\[
b^2 = A^{21}x^1 + A^{22}x^2 = \begin{bmatrix}
60 \\
0
\end{bmatrix} + \begin{bmatrix}
0
\end{bmatrix} = \begin{bmatrix}
60
\end{bmatrix}
\]

we arrive at the corresponding partition for \(b\).
Solving matrix linear equations

Suppose we want to solve a set of linear equations

\[ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \]
\[ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \]
\[ \vdots \]
\[ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \]

or in matrix form

\[ Ax = b. \]

We first put the system into augmented matrix (detached coefficient) form

\[
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} & \mid & b_1 \\
a_{21} & a_{22} & \ldots & a_{2n} & \mid & b_2 \\
\vdots & \vdots & \ddots & \vdots & \mid & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn} & \mid & b_m \\
\end{bmatrix}
\]

**Elementary Row Operations**: The following operations on the rows of a linear system (or the corresponding augmented matrix form) will result in a linear system with the identical set of solutions.

**ero1**: Any row is multiplied by any nonzero scalar.

**ero2**: Any scalar multiple of a row is added to another row.

**ero3**: Any two rows are exchanged.
Example

Consider the system

\[
\begin{align*}
(\text{L}) \quad & 2x_1 + 14x_2 + 12x_3 - 2x_4 = 4 \\
& -x_1 - 7x_2 - 4x_3 - 4x_5 = -1 \\
& x_1 + 7x_2 + 3x_4 + 10x_5 = 2
\end{align*}
\]

Adding 1/2×Row 1 to Row 2 yields equivalent system

\[
\begin{align*}
2x_1 + 14x_2 + 12x_3 - 2x_4 + 2x_3 - x_4 - 4x_5 &= 4 \quad (1) \\
x_1 + 7x_2 + 3x_4 + 10x_5 &= 2
\end{align*}
\]

Adding (-1/2)×Row 1 to Row 3 in the second system yields equivalent system

\[
\begin{align*}
2x_1 + 14x_2 + 12x_3 - 2x_4 + 2x_3 - x_4 - 4x_5 &= 4 \quad (1) \\
x_1 + 7x_2 + 3x_4 + 10x_5 &= 2 \\
6x_3 + 4x_4 + 10x_5 &= 0 \quad (2)
\end{align*}
\]

Finally, multiplying the Row 1 by 1/2 in the third system yields equivalent system

\[
\begin{align*}
x_1 + 7x_2 + 6x_3 - x_4 &= 2 \\
6x_3 + 4x_4 + 10x_5 &= 0 \quad (2)
\end{align*}
\]

The augmented matrix forms are

\[
\begin{bmatrix}
2 & 14 & 12 & -2 & 0 & \vert & 4 \\
-1 & -7 & -4 & 0 & -4 & \vert & -1 \\
1 & 7 & 0 & 3 & 10 & \vert & 2
\end{bmatrix} \quad \rightarrow \quad 
\begin{bmatrix}
2 & 14 & 12 & -2 & 0 & \vert & 4 \\
0 & 0 & 2 & -1 & -4 & \vert & 1 \\
1 & 7 & 0 & 3 & 10 & \vert & 2
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 & 14 & 12 & -2 & 0 & \vert & 4 \\
0 & 0 & 2 & -1 & -4 & \vert & 1 \\
0 & 0 & -6 & 4 & 10 & \vert & 0
\end{bmatrix} \quad \rightarrow \quad 
\begin{bmatrix}
1 & 7 & 6 & -1 & 0 & \vert & 2 \\
0 & 0 & 2 & -1 & -4 & \vert & 1 \\
0 & 0 & -6 & 4 & 10 & \vert & 0
\end{bmatrix}
\]
The Gauss-Jordan Method of Solving Linear Systems

The Gauss-Jordan Method of solving a linear system of \( m \) equations in \( n \) variables uses repeated eros to put the system into the following augmented matrix partitioned form:

\[
\begin{bmatrix}
\bar{A} | \bar{b}
\end{bmatrix} = \begin{bmatrix}
\bar{A}^{11} & \bar{A}^{12} & \bar{b}^1 \\
\bar{A}^{21} & \bar{A}^{22} & \bar{b}^2
\end{bmatrix} = \begin{bmatrix}
I_r & \bar{A}^{12} & \bar{b}^1 \\
0 & 0 & \bar{b}^2
\end{bmatrix}
\]

where \( \bar{A}^{12} \) is an \( r \times (n - r) \) matrix, \( \bar{A}^{21} \) is the \((m - r) \times r \) zero matrix, and \( \bar{A}^{22} \) is the \((m - r) \times (n - r) \) zero matrix. This is called the Gauss-Jordan form for the system.

**Steps of the Gauss-Jordan Method:** Perform the following operations, starting with Row 1 and proceeding downward. Suppose when we get to Row \( i \) we have modified augmented matrix form:

\[
\begin{bmatrix}
\bar{A} | \bar{b}
\end{bmatrix} = \begin{bmatrix}
\bar{a}_{11} & \bar{a}_{12} & \ldots & \bar{a}_{1n} & \bar{b}_1 \\
\bar{a}_{21} & \bar{a}_{22} & \ldots & \bar{a}_{2n} & \bar{b}_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\bar{a}_{m1} & \bar{a}_{m2} & \ldots & \bar{a}_{mn} & \bar{b}_m
\end{bmatrix}
\]

1. Find the first element \( \bar{a}_{ij} \) in Row \( i \) that is nonzero. If all of the \( \bar{A} \) elements in Row \( i \) are zero, go to Row \( i + 1 \), and repeat Step 1.

2. For each row \( k \neq i \), using ero2 add the multiple \(-\bar{a}_{kj}/\bar{a}_{ij}\) of Row \( i \) to Row \( k \).

3. Using ero1, multiply Row \( i \) by \( 1/\bar{a}_{ij} \). Go to Row \( i + 1 \) and repeat at Step 1.

One step of this process is referred to as pivoting on entry \( \bar{a}_{ij} \).
Example

For the system \((L)\)

\[
\begin{align*}
2x_1 + 14x_2 + 12x_3 - 2x_4 &= 4 \\
-x_1 - 7x_2 - 4x_3 - 4x_5 &= -1 \\
x_1 + 7x_2 + 3x_4 + 10x_5 &= 2
\end{align*}
\]

The augmented matrix forms is

\[
\begin{bmatrix}
2 & 14 & 12 & -2 & 0 & 4 \\
-1 & -7 & -4 & 0 & -4 & -1 \\
1 & 7 & 0 & 3 & 10 & 2
\end{bmatrix}
\]

We have already applied the Gauss-Jordan Method to the first row, choosing the element \(a_{11}\) as the first nonzero element in Row 1. After performing the reduction steps, we got system

\[
\begin{bmatrix}
1 & 7 & 6 & -1 & 0 & 2 \\
0 & 0 & 2 & -1 & -4 & 1 \\
0 & 0 & -6 & 4 & 10 & 0
\end{bmatrix}
\]

Now we continue to Row 2, choosing first element \(a_{23}\) as the first nonzero element. After performing the reduction steps, we get system

\[
\begin{bmatrix}
1 & 7 & 0 & 2 & 12 & -1 \\
0 & 0 & 1 & -1/2 & -2 & 1/2 \\
0 & 0 & 0 & 1 & -2 & 3
\end{bmatrix}
\]

Finally, we get to Row 3, choose element \(a_{34}\), and after the reduction steps, get final system

\[
\begin{bmatrix}
1 & 7 & 0 & 0 & 16 & -7 \\
0 & 0 & 1 & 0 & -3 & 2 \\
0 & 0 & 0 & 1 & -2 & 3
\end{bmatrix}
\]
Properties of the Gauss-Jordan Method

1. After Row $i$ is processed, either $\bar{A}_i$ is zero, or there is a column $j$ of $\bar{A}$ which has a 1 in Row $i$, and zeros everywhere else.

2. It follows that after all $m$ rows are processed, by performing the appropriate column and row permutations, we can partition the augmented matrix into the required Gauss-Jordan form.

3. Any rows of the final matrix that are completely zero correspond to redundant equalities of the original system. These equalities are multiples of other equalities in the system, and can be removed without affecting the set of solutions.

4. Any row $i$ whose $\bar{a}_{ij}$ elements are all 0 but whose right-hand-side coefficient $\bar{b}_i \neq 0$ corresponds to an inconsistent equation, that is, one which cannot be satisfied simultaneously with the other equations in the system. The corresponding system therefore has no solution.

Definition: The rank of a matrix $A$ is the number $r$ of nonzero rows of $A$ in the final Gauss-Jordan form. $A$ has full row rank if this rank is equal to the number $m$ of rows of $A$. 
Example

If we change the final equation in the system (L) to

\[ x_1 + 7x_2 + 2x_4 + 12x_5 = -1 \]

then the final augmented matrix form is

\[
\begin{bmatrix}
1 & 7 & 0 & 2 & 12 & -1 \\
0 & 0 & 1 & -1/2 & -2 & 1/2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

indicating that this equation is redundant. If we change
the final equation to

\[ x_1 + 7x_2 + 2x_4 + 12x_5 = 2 \]

then the final augmented matrix form is

\[
\begin{bmatrix}
1 & 7 & 0 & 2 & 12 & -1 \\
0 & 0 & 1 & -1/2 & -2 & 1/2 \\
0 & 0 & 0 & 0 & 0 & 3 \\
\end{bmatrix}
\]

indicating that the final equation is inconsistent, and so
the system has no solution.
Finding Solutions to a Consistent Linear System

Suppose that after the Gauss-Jordan Method is performed on the linear system, there are no inconsistent rows. After deleting all redundant rows we obtain system

\[
\begin{bmatrix} \bar{A} & \bar{b} \end{bmatrix} = \begin{bmatrix} I & \bar{N} & \bar{b} \end{bmatrix}
\]

that is equivalent to the original system. Let the variables corresponding to the columns of \( I \) (in order) be \( x_B = [x_{B_1}, x_{B_2}, \ldots, x_{B_r}]^T \) — called basic (dependent) variables — and those corresponding to \( \bar{N} \) as \( x_N = [x_{N_1}, x_{N_2}, \ldots, x_{N_{n-r}}]^T \) — called nonbasic (independent) variables. Then we have the following system equivalent to the original set of equations:

\[
Ix_B + \bar{N}x_N = \bar{b}, \quad \text{or} \quad x_B = \bar{b} - \bar{N}x_N
\]

**Fact:** The set of solutions to the original system consists of all solutions of the form

\[
\begin{align*}
x_N &= \alpha \\
x_B &= \bar{b} - \bar{N}\alpha
\end{align*}
\]

for any choice of \((n - r)\)-vector \( \alpha \).

**Example:** Using the final form of the tableau for the original system, we have \( x_B = [x_1, x_3, x_4] \) and \( x_N = [x_2, x_5] \). Thus the set of solutions to this system is

\[
x_1 = -7 - 7\alpha_2 - 16\alpha_5, \quad x_2 = \alpha_2, \quad x_3 = 2 + 3\alpha_5, \quad x_4 = 3 + 2\alpha_5, \quad x_5 = \alpha_5
\]

One specific solution, for example, might set \( \alpha_2 = 1 \) and \( \alpha_5 = -1 \), giving resulting solution \( x = [2, 1, -1, 1, -1] \), which clearly satisfies the equations.
Row Operations and Matrix Multiplication

**Fact:** Each of the *eros* actually corresponds to the *multiplication* of the augmented matrix *on the left* by a nonsingular $m \times m$ matrix called the *eta matrix*. (Proof: Exercise).

This means that any set of row operations produces a series of multiplications of the original system on the left by nonsingular matrices:

$$[ \bar{A} \mid \bar{b} ] = E^k \cdots E^2 E^1 [ A \mid b ] = M [ A \mid b ].$$

In fact, you can get this by attaching an identity matrix to the left of the augmented system while you are doing the steps. For system (L):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times [A|b] = \begin{pmatrix} 2 & 14 & 12 & -2 & 0 & 4 \\ -1 & -7 & -4 & 0 & -4 & -1 \\ 1 & 7 & 0 & 3 & 10 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1/2 & 0 & 0 \end{pmatrix} \times [A|b] = \begin{pmatrix} 1 & 7 & 6 & -1 & 0 & 2 \\ 0 & 0 & 2 & -1 & -4 & 1 \\ 0 & 0 & -6 & 4 & 10 & 0 \end{pmatrix}.$$  

$$\begin{pmatrix} -1 & -3 & 0 \\ 1/4 & 1/2 & 0 \\ 1 & 3 & 1 \end{pmatrix} \times [A|b] = \begin{pmatrix} 1 & 7 & 0 & 2 & 12 & -1 \end{pmatrix}, \quad \begin{pmatrix} -3 & -9 & -2 \\ 3/4 & 2 & 1/2 \end{pmatrix} \times [A|b] = \begin{pmatrix} 1 & 7 & 0 & 0 & 16 & -7 \\ 0 & 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 0 & 1 & -2 & 3 \end{pmatrix}.$$  

It follows from the definition of matrix multiplication that if the $i^{th}$ row of $M$ is $[\alpha_1, \ldots, \alpha_m]$, then Row $i$ in the current system is equal to

$$\alpha_1 \times \text{(original Row 1)} + \alpha_2 \times \text{(original Row 2)} + \ldots + \alpha_m \times \text{(original Row m)}$$
Basic Solutions

Suppose $A$ has full row rank.

**Important fact about the $M$ matrix:** Let

$$B = [A_{B_1}, A_{B_2}, \ldots, A_{B_r}]$$

be the submatrix of columns corresponding to the basic variables $x_B = [x_{B_1}, x_{B_2}, \ldots, x_{B_r}]^T$. Then since these columns are reduced to the identity matrix $I_m$ in the final tableau, then from the above discussion it must be that

$$MB = I, \text{ i.e., } M = B^{-1}$$

It follows that if the original matrix is partitioned as $[B \, N \mid b]$, then the final tableau has

$$\bar{N} = B^{-1}N, \quad \bar{b} = B^{-1}b$$

**Fact:** This is true for any choice of basic variables $x_B$ (in any order) so long as the associated submatrix $B$ is nonsingular. Since the systems are clearly equivalent, then the set of solutions of the original system can be described by

$$x_B = \bar{b} - \bar{N}x_N = B^{-1}b - B^{-1}Nx_N$$

One particular solution is obtained by setting

$$x_B = B^{-1}b, \quad x_N = 0.$$  

This is called the basic solution corresponding to basis matrix $B$. 

Example

For system \( \mathbf{L} \), consider basic vector \( x_B = [x_4, x_5, x_1] \). The basis matrix is
\[
B = \begin{bmatrix}
-2 & 0 & 2 \\
0 & -4 & -1 \\
3 & 10 & 1
\end{bmatrix}
\]

which has inverse
\[
B^{-1} = \begin{bmatrix}
1/2 & 5/3 & 2/3 \\
-1/4 & -2/3 & -1/6 \\
1 & 5/3 & 2/3
\end{bmatrix}
\]

Then
\[
\tilde{b} = B^{-1}b = \begin{bmatrix}
1/2 & 5/3 & 2/3 \\
-1/4 & -2/3 & -1/6 \\
1 & 5/3 & 2/3
\end{bmatrix}
\begin{bmatrix}
4 \\
-1 \\
2
\end{bmatrix}
= \begin{bmatrix}
5/3 \\
-2/3 \\
11/3
\end{bmatrix}
\]

\[
\tilde{N} = B^{-1}N = \begin{bmatrix}
1/2 & 5/3 & 2/3 \\
-1/4 & -2/3 & -1/6 \\
1 & 5/3 & 2/3
\end{bmatrix}
\begin{bmatrix}
14 & 12 \\
-7 & -4 \\
7 & 0
\end{bmatrix}
= \begin{bmatrix}
0 & -2/3 \\
0 & -1/3 \\
7 & 16/3
\end{bmatrix}
\]
so that the basic augmented matrix system for this basis is
\[
\begin{bmatrix}
0 & 0 & -2/3 & 1 & 0 & | & 5/3 \\
0 & 0 & -1/3 & 0 & 1 & | & -2/3 \\
1 & 7 & 16/3 & 0 & 0 & | & 11/3
\end{bmatrix}
\]

and the corresponding basic solution is then
\[
x_B = \begin{bmatrix}
x_4 \\
x_5 \\
x_1
\end{bmatrix}
= \begin{bmatrix}
5/3 \\
-2/3 \\
11/3
\end{bmatrix}
\]

\[
x_N = \begin{bmatrix}
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
Determinants and Solutions to Linear Systems

Let $A$ be any $m \times m$ matrix. The $(i, j)^{th}$ minor of $A$ is the $(m - 1) \times (m - 1)$ submatrix $A_{i,j}$ obtained from $A$ by deleting the $i^{th}$ row and $j^{th}$ column of $A$. The determinant of $A$, denoted det $A$, is defined recursively by

$$
\text{det } A = \begin{cases} 
    a_{11}, & m = 1 \\
    \sum_{i=1}^{m} (-1)^{i+j}a_{ij} \text{ det } A_{i,j}, & m > 1
\end{cases}
$$

where the sum can be computed on any column $j$.

Example:

$$
\text{det } \begin{bmatrix}
-2 & 0 & 2 \\
0 & -4 & -1 \\
3 & 10 & 1
\end{bmatrix}
$$

$$
= -2 \text{ det } \begin{bmatrix}
-4 & -1 \\
10 & 1
\end{bmatrix} - 0 \text{ det } \begin{bmatrix}
0 & 2 \\
10 & 1
\end{bmatrix} + 3 \text{ det } \begin{bmatrix}
0 & 2 \\
-4 & -1
\end{bmatrix}
$$

$$
= -2[(-4)(1) - (-1)(10)] + 3[(0)(-1) - (2)(-4)] = 12
$$

Facts:

1. $A$ is **nonsingular** if and only if $\text{det } A \neq 0$.
2. If $A$ and $B$ are two $m \times m$ matrices, then

$$
\text{det}(AB) = \text{det } A \text{ det } B
$$
Cramer’s Rule

Let $A$ be a nonsingular matrix. Then the solution to the system $Ax = b$ is

$$x_j = \frac{\det A|_j}{\det A}, \quad j = 1, \ldots, m$$

where $A|_j$ is the matrix obtained from $A$ by substituting $b$ for the $j^{th}$ column of $A$.

Example: To solve the system

$$\begin{bmatrix}
-2 & 0 & 2 \\
0 & -4 & -1 \\
3 & 10 & 1
\end{bmatrix}\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
4 \\
-1 \\
2
\end{bmatrix}$$

we set

$$A = \begin{bmatrix}
-2 & 0 & 2 \\
0 & -4 & -1 \\
3 & 10 & 1
\end{bmatrix},$$

$$A|_1 = \begin{bmatrix}
4 & 0 & 2 \\
-1 & -4 & -1 \\
2 & 10 & 1
\end{bmatrix}, \quad A|_2 = \begin{bmatrix}
-2 & 4 & 2 \\
0 & -1 & -1 \\
3 & 2 & 1
\end{bmatrix}, \quad A|_3 = \begin{bmatrix}
-2 & 0 & 4 \\
0 & -4 & -1 \\
3 & 10 & 2
\end{bmatrix}.$$

Then

$$\det A = 12, \quad \det A|_1 = 20, \quad \det A|_1 = -8, \quad \det A|_1 = 44,$$

and so

$$x_1 = 20/12, \quad x_2 = -8/12, \quad x_3 = 44/12.$$
Inequalities

For two row or column $n$-vectors $x$ and $y$ we write $x \leq y$ ($x \geq y$) to mean $x_j \leq y_j$ ($x_j \geq y_j$) for all $j = 1, \ldots, n$.

Equality Min LP:

$$
\text{min } z = cx \\
Ax = b \\
x \geq 0
$$

where $x$ is the column $n$-vector of variables, $c$ is the row $n$-vector of objective coefficients, $A$ is the $m \times n$ matrix of proportion coefficients, and $b$ is the column $m$-vector of right-hand-side coefficients.

Inequality Max LP:

$$
\text{max } z = cx \\
Ax \leq b
$$
The Simplex Method
The Simplex Method

We start with the standard equality form LP

$$\min z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$
$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2$$
$$\vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m$$
$$x_1 \geq 0 \quad x_2 \geq 0 \quad \ldots \quad x_n \geq 0.$$  

We will assume that the $A$-matrix has rank $m$. We first put the objective function into equality form:

$$c_1 x_1 + c_2 x_2 + \ldots + c_n x_n - z = 0$$

and then put the whole system into augmented matrix form, with the objective function equality at the bottom.

<table>
<thead>
<tr>
<th>$-z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\ldots$</th>
<th>$x_n$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$a_{11}$</td>
<td>$a_{12}$</td>
<td>$\ldots$</td>
<td>$a_{1n}$</td>
<td>$b_1$</td>
</tr>
<tr>
<td>0</td>
<td>$a_{21}$</td>
<td>$a_{22}$</td>
<td>$\ldots$</td>
<td>$a_{2n}$</td>
<td>$b_2$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>0</td>
<td>$a_{m1}$</td>
<td>$a_{m2}$</td>
<td>$\ldots$</td>
<td>$a_{mn}$</td>
<td>$b_m$</td>
</tr>
<tr>
<td>1</td>
<td>$c_1$</td>
<td>$c_2$</td>
<td>$\ldots$</td>
<td>$c_n$</td>
<td>0</td>
</tr>
</tbody>
</table>

This is called a simplex tableau.
Example

Recall Woody’s LP, in equality form:

\[
\begin{align*}
\text{max } z \\
8x_1 + 12x_2 + x_3 &= 120 \\
15x_2 + x_4 &= 60 \\
3x_1 + 6x_2 + x_5 &= 48 \\
-z - 35x_1 - 60x_2 &= 0 \\
x_1 \geq 0, & & x_2 \geq 0, & & x_3 \geq 0, & & x_4 \geq 0, & & x_5 \geq 0,
\end{align*}
\]

The associated simplex tableau form is

<table>
<thead>
<tr>
<th>(-z)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8</td>
<td>12</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>120</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>15</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>60</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>48</td>
</tr>
<tr>
<td>1</td>
<td>-35</td>
<td>-60</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We can perform row operations on a simplex tableau, just as we did for linear systems, and any solution for a system obtained in this way will satisfy the original set of equalities. It follows that any equivalent tableau in Gauss-Jordan form — called a basic tableau — will produce an associated basic solution which satisfies the original set of equations. In particular, the value of \(z\) — which will always be the negative of the bottom row RHS value — is the objective function value for the associated basic solution.
General Form of a Basic Tableau

Basic simplex tableaus will always have $-z$ as one of the basic variables, and so will look like

$$
\begin{array}{c|cccc|cc}
\text{basis} & -z & x_{B_1} & \ldots & x_{B_m} & x_{N_1} & \ldots & x_{N_{n-m}} \\
\hline
x_{B_1} & 0 & 1 & \ldots & 0 & \bar{a}_{1,N_1} & \ldots & \bar{a}_{1,N_{n-m}} & b_1 \\
& \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x_{B_m} & 0 & 0 & \ldots & 1 & \bar{a}_{m,N_1} & \ldots & \bar{a}_{m,N_{n-m}} & \bar{b}_m \\
-\bar{z} & 1 & 0 & \ldots & 0 & \bar{c}_{N_1} & \ldots & \bar{c}_{N_{n-m}} & -\bar{z}_0 \\
\end{array}
$$

or

$$
\begin{array}{c|ccc|c}
\text{basis} & -z & x_B & x_N & \text{rhs} \\
\hline
x_B & 0 & I & N & b \\
-\bar{z} & 1 & 0 & \bar{c}_N & -\bar{z}_0 \\
\end{array}
$$

**basic feasible solution (BFS):** basic solution that also satisfies the nonnegativity constraints, i.e., such that

$$x_B = \bar{b} \geq 0, \ (x_N = 0)$$

**objective function value of a BFS:** The bottom row equation is

$$-z + \bar{c}_{N_1}x_{N_1} + \ldots + \bar{c}_{N_{n-m}}x_{N_{n-m}} = -\bar{z}_0$$

or

$$z = \bar{z}_0 + \bar{c}_{N_1}x_{N_1} + \ldots + \bar{c}_{N_{n-m}}x_{N_{n-m}}$$

The $\bar{c}_j$ are called the **reduced costs** w.r.t. this basis.
The Phase II Simplex Method

Properties of the Phase II Simplex Method:

iterative method: produce a sequence of solutions $x^1, x^2, \ldots$ that “approach” the optimal solution.

primal method: each of the solutions $x^1, x^2, \ldots$ are feasible to the original LP.

descent/ascent method: the solutions $x^1, x^2, \ldots$ progressively improve the objective function value.

The Phase II Simplex Method assumes that we start with a basic feasible tableau, that is, a basic tableau whose RHS values $\bar{b}$ are all nonnegative. We will produce a sequence of basic feasible solutions with progressively decreasing objective function values, until we reach a solution which minimizes the objective, and hence is optimal to the LP.
Each step of the simplex method involves moving from one BFS along one of a specific set of **pivot directions** which will bring us to another BFS. The directions are obtained by looking at the equation

\[ x_B = \bar{b} - N x_N \]

and choosing a **single** independent variable \( x_j \) — called the **entering variable** — to change. The corresponding equation is

\[ x_B = \bar{b} - \bar{N}_j x_j. \]

Consider changing \( x_j \) from its current value of 0 to a value \( \Delta \geq 0 \). This results in a unique point \( x^\Delta \) defined by

\[
\begin{align*}
\hat{x}_j^\Delta &= \Delta, \\
\hat{x}_{Bi}^\Delta &= \bar{b}_i - \bar{a}_{ij} \Delta, \quad i = 1, \ldots, m. \\
\hat{x}_k^\Delta &= 0, \quad k \neq j, B_1, \ldots, B_m
\end{align*}
\]

Thus for any \( \Delta \), \( x^\Delta \) will satisfy the equation \( Ax^\Delta = b \).

**Effect on the objective function value:** using the equation

\[ z = z_0 + \bar{c}_{N_1} x_{N_1} + \cdots + \bar{c}_{N_{n-m}} x_{N_{n-m}} \]

we get that the objective function value of \( x^\Delta \) is

\[ z^\Delta = z_0 + \bar{c}_j \Delta. \]
Example

Consider the Woody’s Problem starting tableau (with basic variables identified)

<table>
<thead>
<tr>
<th>basis</th>
<th>−z</th>
<th>x₁</th>
<th>x₂</th>
<th>x₃</th>
<th>x₄</th>
<th>x₅</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>x₃</td>
<td>0</td>
<td>8</td>
<td>12</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>120</td>
</tr>
<tr>
<td>x₄</td>
<td>0</td>
<td>0</td>
<td>15</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>60</td>
</tr>
<tr>
<td>x₅</td>
<td>0</td>
<td>3</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>48</td>
</tr>
<tr>
<td>−z</td>
<td>1</td>
<td>−35</td>
<td>−60</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The current BFS has x₁ = 0, x₂ = 0, x₃ = 120, x₄ = 60, x₅ = 48, and objective value z = 0. Suppose we choose x₂ as the entering variable. Then xΔ will have as its component values

\[ x₁^Δ = 0, \]
\[ x₂^Δ = Δ, \]
\[ x₃^Δ = 120 - 12Δ \]
\[ x₄^Δ = 60 - 15Δ \]
\[ x₅^Δ = 48 - 6Δ \]

and its objective value will be −zΔ = 0 + 60Δ, or zΔ = −60Δ.

**Condition for this being an objective-function-decreasing direction:** The reduced cost of the variable must be negative.
Summary of a Simplex Pivot

Choose the entering variable. Choose the variable $x_j$ with a negative reduced cost $\bar{c}_j$ to enter the basis.

Obtain the maximum increase in $\Delta$ for which $x^\Delta$ remains feasible. In order that $x^\Delta$ remains feasible, we must have:

$$x^\Delta_B = \bar{b}_i - \bar{a}_{ij} \Delta \geq 0$$

and so the maximum value $\Delta$ can take on is the minimum ratio

$$\Delta_* = \min \{ \frac{\bar{b}_i}{\bar{a}_{ij}} : \bar{a}_{ij} > 0 \}.$$  

The variable $x_B$, whose index $i$ determines the minimum ratio is called the blocking variable.

Choose the leaving variable. If we set $\Delta = \Delta_*$, then the blocking variable $x_B$ becomes 0, and so $x_B$ leaves the basis.

Perform a simplex pivot: Pivot on the element $\bar{a}_{ij}$, where $x_j$ is the entering variable and $x_B$ is the leaving variable.
Pivoting for Woody’s LP

Starting feasible tableau:

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>8</td>
<td>12</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>120</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0</td>
<td>0</td>
<td>15</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>60</td>
</tr>
<tr>
<td>$x_5$</td>
<td>0</td>
<td>3</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>48</td>
</tr>
<tr>
<td>$-z$</td>
<td>1</td>
<td>-35</td>
<td>-60</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

solution value: $x_1 = 0, x_2 = 0, x_3 = 120, x_4 = 60, x_5 = 48$

objective function value: $z = 0$

entering variable: $x_2$ (objective row coefficient -60)

ratio test:

$$\Delta_* = \min\{120/12, 60/15, 48/6\} = 4$$

leaving variable: $x_4$ (minimum occurred in Row 2)

pivot on entry $\bar{a}_{22}$
New tableau:

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>8</td>
<td>0</td>
<td>1</td>
<td>$-4/5$</td>
<td>0</td>
<td>72</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1/15</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>$x_5$</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>$-2/5$</td>
<td>1</td>
<td>24</td>
</tr>
<tr>
<td>$-z$</td>
<td>1</td>
<td>$-35$</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>240</td>
</tr>
</tbody>
</table>

Solution value: $x_1 = 0, x_2 = 4, x_3 = 72, x_4 = 0, x_5 = 24$

Objective function value: $z = -240$

Notice that the new solution remains feasible and that the objective function value has decreased. We can now start with this basic tableau, and continue to perform simplex pivots, obtaining a sequence of BFSs with decreasing objective function values.

Second Pivot:

entering variable for the new tableau: $x_1$

ratio test:

$$\Delta_* = \min\{72/8, 24/3\} = 8$$

leaving variable: $x_5$ (Row 3)

pivot on entry $\bar{a}_{13}$
New tableau:

<table>
<thead>
<tr>
<th>basis</th>
<th>(-z)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>(\text{rhs})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_3)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>4/15</td>
<td>(-8/3)</td>
<td>8</td>
</tr>
<tr>
<td>(x_2)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1/15</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>(x_1)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>(-2/15)</td>
<td>1/3</td>
<td>8</td>
</tr>
<tr>
<td>(-z)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(-2/3)</td>
<td>35/3</td>
<td>520</td>
</tr>
</tbody>
</table>

Solution value: \(x_1 = 8, x_2 = 4, x_3 = 8, x_4 = 0, x_5 = 0\)

Objective function value: \(z = -520\)

Third Pivot

entering variable: \(x_4\)

ratio test:
\[\Delta_* = \min\{ \frac{8}{4/15}, \frac{4}{1/15} \} = 30\]

leaving variable: \(x_3\) (Row 1)

pivot on entry \(\bar{a}_{14}\)
Fourth tableau:

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>15/4</td>
<td>1</td>
<td>$-10$</td>
<td>30</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$-1/4$</td>
<td>0</td>
<td>2/3</td>
<td>2</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$1/2$</td>
<td>0</td>
<td>$-1$</td>
<td>12</td>
</tr>
<tr>
<td>$-z$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>5/2</td>
<td>0</td>
<td>5</td>
<td>540</td>
</tr>
</tbody>
</table>

Solution value: $x_1 = 12$, $x_2 = 2$, $x_3 = 0$, $x_4 = 30$, $x_5 = 0$

Objective function value: $z = -540$

Stopping Rule #1: If the current tableau has all objective row coefficients nonnegative, STOP, current bfs is optimal to the LP.

Proof: The current BFS has objective function value $z = z_0$. Moreover, the objective function has equivalent form

$$z = \bar{z}_0 + \bar{c}_{N_1}x_{N_1} + \ldots + \bar{c}_{N_{n-m}}x_{N_{n-m}}$$

where all of the $\bar{c}_j$ are nonnegative. This means that every nonnegative solution to the equations $Ax = b$ will have objective function value at least $\bar{z}_0$, and so the current BFS is optimal to the original LP.
Another Example

Recall the unbounded LP given in Lecture 1:

\[
\begin{align*}
\text{max } z &= 35x_1 + 60x_2 \\
&\quad -8x_1 + 12x_2 \leq 120 \\
&\quad -20x_1 + 15x_2 \leq 60 \\
&\quad 3x_1 - 6x_2 \leq 48 \\
&\quad x_1 \geq 0 \quad x_2 \geq 0
\end{align*}
\]

The Phase II pivots for this are \( \square = \text{pivot entry} \)

<table>
<thead>
<tr>
<th>basis</th>
<th>(-z)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_3)</td>
<td>0</td>
<td>-8</td>
<td>12</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>120</td>
</tr>
<tr>
<td>(x_4)</td>
<td>0</td>
<td>-20</td>
<td>15</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>60</td>
</tr>
<tr>
<td>(x_5)</td>
<td>0</td>
<td>3</td>
<td>-6</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>48</td>
</tr>
</tbody>
</table>

\[
\begin{array}{ccccccc}
-1 & -35 & -60 & 0 & 0 & 0 & 0 \\
\end{array}
\]

<table>
<thead>
<tr>
<th>basis</th>
<th>(-z)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1)</td>
<td>0</td>
<td>8</td>
<td>0</td>
<td>1</td>
<td>-4/5</td>
<td>0</td>
<td>72</td>
</tr>
<tr>
<td>(x_4)</td>
<td>0</td>
<td>-4/3</td>
<td>1</td>
<td>0</td>
<td>1/15</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>(x_5)</td>
<td>0</td>
<td>-5</td>
<td>0</td>
<td>0</td>
<td>2/5</td>
<td>1</td>
<td>72</td>
</tr>
</tbody>
</table>

\[
\begin{array}{ccccccc}
1 & -115 & 0 & 0 & 4 & 0 & 240 \\
\end{array}
\]

<table>
<thead>
<tr>
<th>basis</th>
<th>(-z)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1/8</td>
<td>-1/10</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>(x_2)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1/6</td>
<td>-1/15</td>
<td>0</td>
<td>16</td>
</tr>
<tr>
<td>(x_5)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5/8</td>
<td>-1/10</td>
<td>1</td>
<td>117</td>
</tr>
</tbody>
</table>

\[
\begin{array}{ccccccc}
1 & 0 & 0 & 115/8 & -15/8 & 0 & 1275 \\
\end{array}
\]
**Stopping Rule #2:** If there exists a variable $x_j$ with $\bar{c}_j < 0$, but for which $\bar{a}_{ij} \leq 0$, $i = 1, \ldots, m$, STOP, the LP is unbounded.

**Proof:** The set of solutions $x^\Delta$ has

$$
\begin{align*}
x_j^\Delta &= \Delta, \\
x_{B_i}^\Delta &= \bar{b}_i - \bar{a}_{ij}\Delta, \quad i = 1, \ldots, m. \\
x_k^\Delta &= 0, \quad k \neq j, B_1, \ldots, B_m
\end{align*}
$$

and so $x^\Delta$ will be feasible for *every* choice of $\Delta \geq 0$. But the objective value for $x^\Delta$ is

$$z^\Delta = \bar{z}_0 + \bar{c}_j\Delta,$$

which can be made arbitrarily small for large enough choice of $\Delta$.

In our example, an attempted pivot in Column 4 results in the unbounded set of solutions

$$
\begin{align*}
x_1^\Delta &= 9 + \Delta/10, \quad x_2^\Delta = 16 + \Delta/15 \\
x_3^\Delta &= \Delta, \quad x_4^\Delta = 0, \quad x_5^\Delta = 117 + \Delta/10
\end{align*}
$$

with objective values

$$z^\Delta = -1275 - 15\Delta/8$$

which goes to $-\infty$ as $\Delta \to \infty$. 
Summary of the Phase II Simplex Method

Beginning with **basic feasible tableau**

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$x_{B_1}$</th>
<th>...</th>
<th>$x_{B_m}$</th>
<th>$x_{N_1}$</th>
<th>...</th>
<th>$x_{N_{n-m}}$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{B_1}$</td>
<td>0</td>
<td>1</td>
<td>...</td>
<td>0</td>
<td>$\bar{a}_{1,N_1}$</td>
<td>...</td>
<td>$\bar{a}<em>{1,N</em>{n-m}}$</td>
<td>$b_1 \geq 0$</td>
</tr>
<tr>
<td>...</td>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>$x_{B_m}$</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>1</td>
<td>$\bar{a}_{m,N_1}$</td>
<td>...</td>
<td>$\bar{a}<em>{m,N</em>{n-m}}$</td>
<td>$\bar{b}_m \geq 0$</td>
</tr>
<tr>
<td>$-z$</td>
<td>1</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>$\bar{c}_{N_1}$</td>
<td>...</td>
<td>$\bar{c}<em>{N</em>{n-m}}$</td>
<td>$-\bar{z}_0$</td>
</tr>
</tbody>
</table>

repeat the following steps until one of the stopping rules is satisfied:

1. Choose as **entering variable** any $x_j$ having $\bar{c}_j < 0$.
   (Heuristic best choice: **minimum** $\bar{c}_j$.) If all $\bar{c}_j \geq 0$, **STOP**, the current BFS is **optimal**.

2. Choose as **leaving variable** that variable $x_{B_i}$ such that

   $$\bar{b}_i/\bar{a}_{ij} = \min\{\bar{b}_k/\bar{a}_{kj} \mid \bar{a}_{kj} > 0, \ k = 1, \ldots, m\}$$

   If all $\bar{a}_{kj} \leq 0, \ k = 1, \ldots, m$, then **STOP**, the LP is **unbounded**.

3. Perform a pivot on entry $\bar{a}_{ij}$. 
Finding an Initial Starting Basic Feasible Tableau

Generally we will not be able to start with a basic feasible tableau. For example, consider the following LP:

\[
\begin{align*}
\text{min } z &= -15x_1 + 12x_2 + 10x_3 + 23x_4 - 24x_5 - 11x_6 + 48x_7 \\
4x_1 - 3x_2 - 2x_3 - 6x_4 + 8x_5 + 3x_6 - 9x_7 &= 164 \\
-2x_1 + x_2 + 4x_3 + 4x_4 + 2x_5 - x_6 + x_7 &= -66 \\
12x_1 - 4x_2 - 6x_3 - 13x_4 + 9x_5 + 4x_6 - 7x_7 &= 272
\end{align*}
\]

We first put it into a simplex tableau form:

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>?</td>
<td>0</td>
<td>4</td>
<td>-3</td>
<td>-2</td>
<td>-6</td>
<td>8</td>
<td>3</td>
<td>-9</td>
<td>164</td>
</tr>
<tr>
<td>?</td>
<td>0</td>
<td>-2</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>-66</td>
</tr>
<tr>
<td>?</td>
<td>0</td>
<td>12</td>
<td>-4</td>
<td>-6</td>
<td>-13</td>
<td>9</td>
<td>4</td>
<td>-7</td>
<td>272</td>
</tr>
<tr>
<td>$-z$</td>
<td>1</td>
<td>-15</td>
<td>12</td>
<td>10</td>
<td>23</td>
<td>-24</td>
<td>-11</td>
<td>48</td>
<td>0</td>
</tr>
</tbody>
</table>

We need to find a basic tableau for this system which in addition has nonnegative RHS. To do this we will perform a series of applications of the Phase II Simplex Method to find a basis element for each row while maintaining nonnegative RHSs for the rows already processed.
The Row-Progressive Phase I Simplex Method

We will always be dealing with a “partially feasible” basic tableau that has nonnegative RHS elements in its first $r$ rows:

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$ $x_{B_1}$ $x_{B_2}$ $x_{B_{r+1}}$ $x_{B_m}$ $x_{N_1}$ $x_{N_{n-m}}$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{B_1}$</td>
<td>$0$ $1$ $0$ $0$ $0$ $0$ $0$ $\bar{a}<em>{1,N_1}$ $\cdots$ $\bar{a}</em>{1,N_{n-m}}$</td>
<td>$b_1 \geq 0$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$ $\ddots$ $\vdots$ $\ddots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$x_{B_r}$</td>
<td>$0$ $0$ $1$ $0$ $0$ $0$ $\bar{a}<em>{r,N_1}$ $\cdots$ $\bar{a}</em>{r,N_{n-r}}$</td>
<td>$\bar{b}_r \geq 0$</td>
</tr>
<tr>
<td>$x_{B_{r+1}}$</td>
<td>$0$ $0$ $0$ $1$ $0$ $0$ $\bar{a}<em>{r+1,N_1}$ $\cdots$ $\bar{a}</em>{r+1,N_{n-m}}$</td>
<td>$\bar{b}_{r+1}$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$ $\ddots$ $\vdots$ $\ddots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$x_{B_m}$</td>
<td>$0$ $0$ $0$ $0$ $1$ $\bar{a}<em>{m,N_1}$ $\cdots$ $\bar{a}</em>{m,N_{n-m}}$</td>
<td>$\bar{b}_m$</td>
</tr>
<tr>
<td>$-z$</td>
<td>$1$ $0$ $0$ $0$ $0$ $0$ $\bar{c}<em>{N_1}$ $\cdots$ $\bar{c}</em>{N_{n-m}}$</td>
<td>$-\bar{z}_0$</td>
</tr>
</tbody>
</table>

**Initial Step:** Perform a standard Gauss-Jordan reduction of the original equality system. If a redundant row is found delete it, and if an inconsistent row is found **STOP**, the LP cannot have a feasible solution (since the equality system cannot have any solution).

In our case, we pivot on the first three columns to produce basic tableau:

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$ $x_1$ $x_2$ $x_3$ $x_4$ $x_5$ $x_6$ $x_7$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$0$ $1$ $0$ $0$ $-1/2$ $1/2$ $0$ $1/2$</td>
<td>$7$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$0$ $0$ $1$ $0$ $1$ $-3$ $-1$ $4$</td>
<td>$-44$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$0$ $0$ $0$ $1$ $1/2$ $3/2$ $0$ $-1/2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$-z$</td>
<td>$1$ $0$ $0$ $0$ $-3/2$ $9/2$ $1$ $-5/2$</td>
<td>$653$</td>
</tr>
</tbody>
</table>
Iterative Step:

For each row \( r = 1, \ldots, m \) perform the following:

1. If the RHS element \( \bar{b}_r \) is nonnegative, then Row \( r \) is in the correct form to be included in the set of rows with nonnegative RHSs. Go on to Row \( r + 1 \).

2. If \( \bar{b}_r < 0 \), then we need to increase this value until (hopefully) it becomes nonnegative. We do this by treating the Row \( r \) as an objective row, and we perform the simplex method using only the first \( r - 1 \) rows as the \( A \) matrix. This insure a starting feasible tableau (ignoring rows \( r + 1, \ldots, m \)) and the simplex method will proceed to increase the RHS value.

3. If the simplex method succeeds in making the RHS value \( \bar{b}_r \) nonnegative, then we are finished processing Row \( r \). Go on to Row \( r + 1 \).

4. If the simplex method results in an unbounded tableau, then we have a column \( j \) with \( \bar{a}_{rj} < 0 \) and \( \bar{a}_{ij} \leq 0 \) for \( i = 1, \ldots, r - 1 \). Now pivot on element \( \bar{a}_{rj} \). This involves
   - subtracting \( \bar{a}_{ij}/\bar{a}_{rj} \geq 0 \) times Row \( r \) from Row \( i \), which means that the new RHS value for Row \( i \) will be \( \bar{b}_i - (\bar{a}_{ij}/\bar{a}_{rj})\bar{b}_r \geq 0 \). Thus the first \( r - 1 \) rows will continue to have nonnegative RHS values.
   - dividing Row \( r \) by \( \bar{a}_{rj} < 0 \), which means that the new RHS value for Row \( r \) is \( \bar{b}_r/\bar{a}_{rj} > 0 \). Thus Row \( r \) now has nonnegative RHS value. Go on to Row \( r + 1 \).

5. When all of the rows of \( A \) have been successfully processed, the tableau is in basic feasible form, and we can continue Phase II using the actual objective row.
Example

Having put our basic tableau

<table>
<thead>
<tr>
<th>basis</th>
<th>(-z)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>(x_6)</th>
<th>(x_7)</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>(-1/2)</td>
<td>1/2</td>
<td>0</td>
<td>1/2</td>
<td>7</td>
</tr>
<tr>
<td>(x_2)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>(-3)</td>
<td>(-1)</td>
<td>4</td>
<td>(-44)</td>
</tr>
<tr>
<td>(x_3)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>(1/2)</td>
<td>3/2</td>
<td>0</td>
<td>(-1/2)</td>
<td>(-2)</td>
</tr>
<tr>
<td>(-z)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(-3/2)</td>
<td>9/2</td>
<td>1</td>
<td>(-5/2)</td>
<td>653</td>
</tr>
</tbody>
</table>

We proceed to process the rows in order.

**Row 1:** Since this row already has nonnegative RHS, we go on to Row 2.

**Row 2:** \(\bar{b}_2 < 0\), so we apply the simplex method using Row 1 as the \(A\) matrix and Row 2 as the objective row. Then \(x_5\) is the entering variable, and we pivot on element \(\bar{a}_{15} = 1/2\) to produce tableau

<table>
<thead>
<tr>
<th>basis</th>
<th>(-z)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>(x_6)</th>
<th>(x_7)</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_5)</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>(-1)</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>14</td>
</tr>
<tr>
<td>(x_2)</td>
<td>0</td>
<td>6</td>
<td>1</td>
<td>0</td>
<td>(-2)</td>
<td>0</td>
<td>(-1)</td>
<td>7</td>
<td>(-2)</td>
</tr>
<tr>
<td>(x_3)</td>
<td>0</td>
<td>(-3)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>(-2)</td>
<td>(-23)</td>
</tr>
<tr>
<td>(-z)</td>
<td>1</td>
<td>(-9)</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>8</td>
<td>590</td>
</tr>
</tbody>
</table>

The next entering variable is \(x_4\), and here we have an unbounded partial tableau. We pivot on \(\bar{a}_{24} = -2\) to produce tableau
putting the first two rows in the correct form.

Row 3: This also has $\bar{b}_3 < 0$ so we apply the simplex method using Rows 1 and 2 as the $A$ matrix and Row 3 as the objective row. Then $x_6$ is the entering variable, and we pivot on element $\bar{a}_{26} = 1/2$ to produce tableau

$$
\begin{array}{rrrrrrrrr}
\text{basis} & -z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{rhs} \\
x_5 & 0 & -1 & -1/2 & 0 & 0 & 1 & 1/2 & -5/2 & 15 \\
x_4 & 0 & -3 & -1/2 & 0 & 1 & 0 & 1/2 & -7/2 & 1 \\
x_3 & 0 & 3 & 1 & 1 & 0 & 0 & -1 & 5 & -25 \\
-z & 1 & 0 & 3/2 & 0 & 0 & 0 & -1/2 & 37/2 & 587 \\
\end{array}
$$

Now $x_1$ is the entering variable, and we pivot on element $\bar{a}_{11} = 2$ to produce tableau

$$
\begin{array}{rrrrrrrrr}
\text{basis} & -z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{rhs} \\
x_5 & 0 & 2 & 0 & 0 & -1 & 1 & 0 & 1 & 14 \\
x_6 & 0 & -6 & -1 & 0 & 2 & 0 & 1 & -7 & 2 \\
x_3 & 0 & -3 & 0 & 1 & 2 & 0 & 0 & -2 & -23 \\
-z & 1 & -3 & 1 & 0 & 1 & 0 & 0 & 15 & 588 \\
\end{array}
$$
Finally, $x_7$ is the entering variable, and we pivot on element $\bar{a}_{17} = 1/2$ to produce tableau

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0</td>
<td>$-1/2$</td>
<td>$1/2$</td>
<td>0</td>
<td>$1/2$</td>
<td>7</td>
</tr>
<tr>
<td>$x_6$</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
<td>$-1$</td>
<td>3</td>
<td>1</td>
<td>$-4$</td>
<td>44</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$1/2$</td>
<td>$3/2$</td>
<td>0</td>
<td>$-1/2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$-z$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$-1/2$</td>
<td>$3/2$</td>
<td>0</td>
<td>$33/2$</td>
<td>609</td>
</tr>
</tbody>
</table>

This is a basic feasible tableau, and this ends Phase I.

**Phase II:** $x_1$ is the entering variable, a pivot on element $\bar{a}_{31} = 1$ to produce optimal tableau

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_7$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>14</td>
</tr>
<tr>
<td>$x_6$</td>
<td>0</td>
<td>8</td>
<td>$-1$</td>
<td>0</td>
<td>$-5$</td>
<td>7</td>
<td>1</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>$-z$</td>
<td>1</td>
<td>$-33$</td>
<td>1</td>
<td>0</td>
<td>16</td>
<td>$-15$</td>
<td>0</td>
<td>0</td>
<td>378</td>
</tr>
</tbody>
</table>


Detecting Infeasible LPs

Suppose we start with the following LP:

\[
\begin{align*}
\min z &= -15x_1 + 12x_2 + 10x_3 + 23x_4 - 24x_5 - 11x_6 + 48x_7 \\
4x_1 - 3x_2 - 2x_3 - 6x_4 + 8x_5 + 3x_6 - 9x_7 &= 184 \\
-2x_1 + x_2 + 4x_3 + 4x_4 + 2x_5 - x_6 + x_7 &= -106 \\
12x_1 - 4x_2 - 6x_3 - 13x_4 + 9x_5 + 4x_6 - 7x_7 &= 332
\end{align*}
\]

Again pivoting on the first three columns, we get the following basic tableau:

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1/2</td>
<td>1/2</td>
<td>0</td>
<td>1/2</td>
<td>7</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-3</td>
<td>-1</td>
<td>4</td>
<td>-44</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1/2</td>
<td>3/2</td>
<td>0</td>
<td>-1/2</td>
<td>-12</td>
</tr>
<tr>
<td>$-z$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-3/2</td>
<td>9/2</td>
<td>1</td>
<td>-5/2</td>
<td>653</td>
</tr>
</tbody>
</table>

We then take the same first five pivots as the last example, except now we arrive at tableau

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_7$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>14</td>
</tr>
<tr>
<td>$x_6$</td>
<td>0</td>
<td>8</td>
<td>-1</td>
<td>0</td>
<td>-5</td>
<td>7</td>
<td>1</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>-5</td>
</tr>
<tr>
<td>$-z$</td>
<td>1</td>
<td>-33</td>
<td>1</td>
<td>0</td>
<td>16</td>
<td>-15</td>
<td>0</td>
<td>0</td>
<td>378</td>
</tr>
</tbody>
</table>
We can no longer process Row 3 because all of its $A$-matrix entries are nonnegative, but the RHS value is still negative.

**Stopping Rule #3:** If in the Phase I procedure a row $r$ is encountered which has $\bar{a}_{rj} \geq 0$, $j = 1, \ldots, n$ and $\bar{b}_r < 0$, STOP, the LP is infeasible.

**Proof:** The associated equivalent equation for Row $r$ is

$$\bar{a}_{r1}x_1 + \ldots + \bar{a}_{rn}x_n = \bar{b}_r < 0.$$ 

But since the LHS must be nonnegative for any choice of nonnegative $x$, then clearly this equality can never be satisfied for any feasible solution to the LP.

**Summary of Stopping Rules for the LP**

The Phase I-Phase II Simplex Method can stop only upon encountering one of the three stopping rules. Therefore it detects one of three types of LPs:

**Stopping Rule #1:** An optimal solution has been found.

**Stopping Rule #2:** The LP is unbounded.

**Stopping Rule #3:** The LP is infeasible.
The Artificial Phase I Method
and
The Revised Simplex Method
The Artificial Variable Phase I Method

Consider the problem of finding an initial basic feasible tableau for an LP in standard equality form:

\[
\begin{align*}
\min \ z &= c_1 x_1 + \ldots + c_n x_n \\
 a_{11} x_1 + \ldots + a_{1n} x_n &= b_1 \\
&\quad \vdots \\
 a_{m1} x_1 + \ldots + a_{mn} x_n &= b_m \\
x_1 \geq 0 &\quad \ldots \quad x_n \geq 0
\end{align*}
\]

A more efficient way to obtain a starting tableau is the **Artificial Phase I Simplex Method**. To perform this method, we first make all of the RHSs nonnegative, by multiplying each row with negative RHS by -1:

\[
\begin{align*}
\bar{a}_{11} x_1 + \ldots + \bar{a}_{1n} x_n &= \bar{b}_1 \geq 0 \\
&\quad \vdots \\
\bar{a}_{m1} x_1 + \ldots + \bar{a}_{mn} x_n &= \bar{b}_m \geq 0 \\
x_1 \geq 0 &\quad \ldots \quad x_n \geq 0
\end{align*}
\]

We then add **artificial variables** to each row of the tableau:

\[
\begin{align*}
\bar{a}_{11} x_1 + \ldots + \bar{a}_{1n} x_n + x_1^a &= \bar{b}_1 \\
&\quad \vdots \\
\bar{a}_{m1} x_1 + \ldots + \bar{a}_{mn} x_n + x_m^a &= \bar{b}_m \\
x_1 \geq 0, &\quad \ldots \quad x_n \geq 0, \quad x_1^a \geq 0, \ldots \quad x_m^a \geq 0
\end{align*}
\]

By using the set of artificial variables as a starting basis, we obtain a starting basic feasible solution.
A feasible solution to the original problem, however, requires that all of the artificial variables equal 0. To do this we solve a Phase I LP which minimizes the sum of the artificial variables:

\[
\begin{align*}
\min w = & \quad x_1^a + \ldots + x_m^a \\
& a_{11}x_1 + \ldots + a_{1n}x_n + x_1^a = b_1 \\
& \vdots \\
& a_{m1}x_1 + \ldots + a_{mn}x_n + x_m^a = b_m \\
x_1 \geq 0, \ldots, x_n \geq 0, \quad x_1^a \geq 0, \ldots, x_m^a \geq 0
\end{align*}
\]

This LP always has an optimal solution. (Why?) If its optimal value \( w^* = 0 \), then the values of the nonartificial variables will be a feasible solution to the original LP. If it has optimal value \( w^* > 0 \), then the LP is infeasible.

**Example:** Consider the problem

\[
\begin{align*}
\min z = & \quad -15x_1 + 12x_2 + 10x_3 + 23x_4 - 24x_5 - 11x_6 + 48x_7 \\
& 4x_1 - 3x_2 - 2x_3 - 6x_4 + 8x_5 + 3x_6 - 9x_7 = 164 \\
& -2x_1 + x_2 + 4x_3 + 4x_4 + 2x_5 - x_6 + x_7 = -66 \\
& 12x_1 - 4x_2 - 6x_3 - 13x_4 + 9x_5 + 4x_6 - 7x_7 = 272 \\
x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0, \quad x_5 \geq 0, \quad x_6 \geq 0, \quad x_7 \geq 0
\end{align*}
\]

After negating the second row and adding artificial variables, we obtain Phase I LP

\[
\begin{align*}
\min w = & \quad x_1^a + x_2^a + x_3^a \\
& 4x_1 - 3x_2 - 2x_3 - 6x_4 + 8x_5 + 3x_6 - 9x_7 + x_1^a = 164 \\
& 2x_1 - x_2 - 4x_3 - 4x_4 - 2x_5 + x_6 - x_7 + x_2^a = 66 \\
& 12x_1 - 4x_2 - 6x_3 - 13x_4 + 9x_5 + 4x_6 - 7x_7 + x_3^a = 272 \\
x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0, \quad x_5 \geq 0, \quad x_6 \geq 0, \quad x_7 \geq 0, \quad x_1^a \geq 0, \quad x_2^a \geq 0, \quad x_3^a \geq 0
\end{align*}
\]
Tableaus for the Artificial LP

An artificial LP will have two objective function rows, one for the artificial LP and one for the original LP.

We first put the tableau into basic form by costing out the $w$ row, that is, subtracting each of the first $m$ rows from the $w$ row:

We then proceed to minimize the $w$-objective (Phase I). If a feasible solution is found, then the $w$ row and all artificial columns are dropped, and we proceed to optimize the $z$-objective (Phase II). If a tableau is reached where the $w$ row reduced costs are all nonnegative, and the RHS value is still negative, the LP is infeasible.
Example

The artificial tableau for our example problem is

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$-w$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_1^a$</th>
<th>$x_2^a$</th>
<th>$x_3^a$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1^a$</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>-3</td>
<td>-2</td>
<td>-6</td>
<td>8</td>
<td>3</td>
<td>-9</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>164</td>
</tr>
<tr>
<td>$x_2^a$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td>-4</td>
<td>-4</td>
<td>-2</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>66</td>
</tr>
<tr>
<td>$x_3^a$</td>
<td>0</td>
<td>0</td>
<td>12</td>
<td>-4</td>
<td>-6</td>
<td>-13</td>
<td>9</td>
<td>4</td>
<td>-7</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>272</td>
</tr>
<tr>
<td>$-w$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>1</td>
<td>1</td>
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<td>0</td>
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<tr>
<td>$-z$</td>
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<td>0</td>
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<td>10</td>
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<td>-11</td>
<td>48</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

After costing out the $w$ row, we get basic feasible artificial tableau

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$-w$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_1^a$</th>
<th>$x_2^a$</th>
<th>$x_3^a$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1^a$</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>-3</td>
<td>-2</td>
<td>-6</td>
<td>8</td>
<td>3</td>
<td>-9</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>164</td>
</tr>
<tr>
<td>$x_2^a$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td>-4</td>
<td>-4</td>
<td>-2</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>66</td>
</tr>
<tr>
<td>$x_3^a$</td>
<td>0</td>
<td>0</td>
<td>12</td>
<td>-4</td>
<td>-6</td>
<td>-13</td>
<td>9</td>
<td>4</td>
<td>-7</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>272</td>
</tr>
<tr>
<td>$-w$</td>
<td>0</td>
<td>1</td>
<td>-18</td>
<td>8</td>
<td>12</td>
<td>23</td>
<td>-15</td>
<td>-8</td>
<td>17</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-502</td>
</tr>
<tr>
<td>$-z$</td>
<td>1</td>
<td>0</td>
<td>-15</td>
<td>12</td>
<td>10</td>
<td>23</td>
<td>-24</td>
<td>-11</td>
<td>48</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Optimizing with respect to the $w$ row, we pivot first on the $x_1$ column, third row:

$$\begin{array}{cccccccccccccccccccccccccccc} \text{basis} & -z & -w & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_1^a & x_2^a & x_3^a & \text{rhs} \\
 x_1^a & 0 & 0 & 0 & -5/3 & 0 & -5/3 & 5 & 5/3 & -20/3 & 1 & 0 & -1/3 & 220/3 \\
x_2^a & 0 & 0 & 0 & -1/3 & -3 & -11/6 & -7/2 & 1/3 & 1/6 & 0 & 1 & -1/6 & 62/3 \\
x_1 & 0 & 0 & 1 & -1/3 & -1/2 & -13/12 & 3/4 & 1/3 & -7/12 & 0 & 0 & 1/12 & 68/3 \\
-w & 0 & 1 & 0 & 2 & 3 & 7/2 & -3/2 & -2 & 13/2 & 0 & 0 & 3/2 & -94 \\
-z & 1 & 0 & 0 & 7 & 5/2 & 27/4 & -51/4 & -6 & 157/4 & 0 & 0 & 5/4 & 340 \\
\end{array}$$

then in the $x_6$ column, first row:

$$\begin{array}{cccccccccccccccccccccccccccc} \text{basis} & -z & -w & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_1^a & x_2^a & x_3^a & \text{rhs} \\
 x_6 & 0 & 0 & 0 & -1 & 0 & -1 & 3 & 1 & -4 & 3/5 & 0 & -1/5 & 44 \\
x_1^a & 0 & 0 & 0 & 0 & -3 & -3/2 & -9/2 & 0 & 3/2 & -1/5 & 1 & -1/10 & 6 \\
x_1 & 0 & 0 & 1 & 0 & -1/2 & -3/4 & -1/4 & 0 & 3/4 & -1/5 & 0 & 3/20 & 8 \\
-w & 0 & 1 & 0 & 0 & 3 & 3/2 & 9/2 & 0 & -3/2 & 6/5 & 0 & 11/10 & -6 \\
-z & 1 & 0 & 0 & 1 & 5/2 & 3/4 & 21/4 & 0 & 61/4 & 18/5 & 0 & 1/20 & 604 \\
\end{array}$$

and finally, on the $x_7$ column, second row:

$$\begin{array}{cccccccccccccccccccccccccccc} \text{basis} & -z & -w & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_1^a & x_2^a & x_3^a & \text{rhs} \\
 x_6 & 0 & 0 & 0 & -1 & -8 & -5 & -9 & 1 & 0 & 1/15 & 8/3 & -7/15 & 60 \\
x_7 & 0 & 0 & 0 & 0 & -2 & -1 & -3 & 0 & 1 & -2/15 & 2/3 & -1/15 & 4 \\
x_1 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 0 & -1/10 & -1/2 & 1/5 & 5 \\
-w & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
-z & 1 & 0 & 0 & 1 & 33 & 16 & 51 & 0 & 0 & 169/30 & -61/6 & 16/15 & 543 \\
\end{array}$$

The $w$ objective is now minimized at 0, and the basis is now feasible, with feasible solution $x_1 = 5$, $x_2 = x_3 = x_4 = x_5 = 0$, $x_6 = 60$, and $x_7 = 4$. Dropping the $w$ row and the three artificial columns, we see that the current solution is also optimal, and so Phase II has produced an optimal tableau.
If we change the original RHS of the original LP to 184, -106, and 332, we get starting basic feasible tableau

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$-w$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_1^a$</th>
<th>$x_2^a$</th>
<th>$x_3^a$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1^a$</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>-3</td>
<td>-2</td>
<td>-6</td>
<td>8</td>
<td>3</td>
<td>9</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>184</td>
</tr>
<tr>
<td>$x_2^a$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td>-4</td>
<td>-4</td>
<td>-2</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>106</td>
</tr>
<tr>
<td>$x_3^a$</td>
<td>0</td>
<td>0</td>
<td>12</td>
<td>-4</td>
<td>-6</td>
<td>-13</td>
<td>9</td>
<td>4</td>
<td>-7</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>332</td>
</tr>
<tr>
<td>$-w$</td>
<td>0</td>
<td>1</td>
<td>-18</td>
<td>8</td>
<td>12</td>
<td>23</td>
<td>-15</td>
<td>-8</td>
<td>17</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-622</td>
</tr>
<tr>
<td>$-z$</td>
<td>1</td>
<td>0</td>
<td>-15</td>
<td>12</td>
<td>10</td>
<td>23</td>
<td>-24</td>
<td>-11</td>
<td>48</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We take the same first two pivots, resulting in tableau

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$-w$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_1^a$</th>
<th>$x_2^a$</th>
<th>$x_3^a$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>3</td>
<td>1</td>
<td>-4</td>
<td>3/5</td>
<td>0</td>
<td>-1/5</td>
<td>44</td>
</tr>
<tr>
<td>$x_2^a$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-3</td>
<td>-3/2</td>
<td>-9/2</td>
<td>0</td>
<td>3/2</td>
<td>-1/5</td>
<td>1</td>
<td>-1/10</td>
<td>36</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1/2</td>
<td>-3/4</td>
<td>-1/4</td>
<td>0</td>
<td>3/4</td>
<td>-1/5</td>
<td>0</td>
<td>3/20</td>
<td>13</td>
</tr>
<tr>
<td>$-w$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3/2</td>
<td>9/2</td>
<td>0</td>
<td>-3/2</td>
<td>6/5</td>
<td>0</td>
<td>11/10</td>
<td>-36</td>
</tr>
<tr>
<td>$-z$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>5/2</td>
<td>3/4</td>
<td>21/4</td>
<td>0</td>
<td>61/4</td>
<td>18/5</td>
<td>0</td>
<td>1/20</td>
<td>679</td>
</tr>
</tbody>
</table>

Now a pivot in the $x_7$ column must be done in the third row, resulting in tableau

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$-w$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_1^a$</th>
<th>$x_2^a$</th>
<th>$x_3^a$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_6$</td>
<td>0</td>
<td>0</td>
<td>16/3</td>
<td>-1</td>
<td>-8/3</td>
<td>-5</td>
<td>5/3</td>
<td>1</td>
<td>0</td>
<td>-7/15</td>
<td>0</td>
<td>3/5</td>
<td>340/3</td>
</tr>
<tr>
<td>$x_2^a$</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>-4</td>
<td>0</td>
<td>0</td>
<td>1/5</td>
<td>1</td>
<td>-2/5</td>
<td>10</td>
</tr>
<tr>
<td>$x_7$</td>
<td>0</td>
<td>0</td>
<td>4/3</td>
<td>0</td>
<td>-2/3</td>
<td>-1</td>
<td>-1/3</td>
<td>0</td>
<td>1</td>
<td>-4/15</td>
<td>0</td>
<td>1/5</td>
<td>52/3</td>
</tr>
<tr>
<td>$-w$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4/5</td>
<td>0</td>
<td>7/5</td>
<td>-10</td>
</tr>
<tr>
<td>$-z$</td>
<td>1</td>
<td>0</td>
<td>-61/3</td>
<td>1</td>
<td>38/3</td>
<td>16</td>
<td>31/3</td>
<td>0</td>
<td>0</td>
<td>23/3</td>
<td>0</td>
<td>-3</td>
<td>1244/3</td>
</tr>
</tbody>
</table>

The optimal value of $w$ is $10 > 0$, and so the original LP is infeasible.
Matrix Form of a Basic Tableau

Consider a general equality form LP tableau:

\[
\begin{array}{ccc|c}
-z & x_1 & x_2 & \ldots & x_n & \text{rhs} \\
0 & a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \\
1 & c_1 & c_2 & \ldots & c_n & 0
\end{array}
\]

or

\[
\begin{array}{c|c|c}
-z & x & \text{rhs} \\
0 & A & b \\
1 & c & 0
\end{array}
\]

An arbitrary basic tableau will look like

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$x_{B_1}$</th>
<th>$\ldots$</th>
<th>$x_{B_m}$</th>
<th>$x_{N_1}$</th>
<th>$\ldots$</th>
<th>$x_{N_{n-m}}$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{B_1}$</td>
<td>0</td>
<td>1</td>
<td>$\ldots$</td>
<td>0</td>
<td>$\bar{a}_{1,N_1}$</td>
<td>$\ldots$</td>
<td>$\bar{a}<em>{1,N</em>{n-m}}$</td>
<td>$b_1$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\ddots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>\ldots</td>
<td>$\vdots$</td>
<td></td>
</tr>
<tr>
<td>$x_{B_m}$</td>
<td>0</td>
<td>0</td>
<td>$\ldots$</td>
<td>1</td>
<td>$\bar{a}_{m,N_1}$</td>
<td>$\ldots$</td>
<td>$\bar{a}<em>{m,N</em>{n-m}}$</td>
<td>$\bar{b}_m$</td>
</tr>
<tr>
<td>$-z$</td>
<td>1</td>
<td>0</td>
<td>$\ldots$</td>
<td>0</td>
<td>$\bar{c}_{N_1}$</td>
<td>$\ldots$</td>
<td>$\bar{c}<em>{N</em>{n-m}}$</td>
<td>$-\bar{z}_0$</td>
</tr>
</tbody>
</table>

or

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$x_{B}$</th>
<th>$x_{N}$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{B}$</td>
<td>0</td>
<td>$I$</td>
<td>$N$</td>
<td>$b$</td>
</tr>
<tr>
<td>$-z$</td>
<td>1</td>
<td>0</td>
<td>$\bar{c}_{N}$</td>
<td>$-\bar{z}_0$</td>
</tr>
</tbody>
</table>

We have already described the $\bar{N}$ matrix and $\bar{b}$ vector, using the equation

\[
Bx_B + Nx_N = b \quad \text{or} \quad x_B = B^{-1}b - B^{-1}N
\]
How about $\bar{c}_N$ and $-\bar{z}_0$? Writing $z = c_B x_B + c_N x_N$ and substituting for $x_B$, we get

$$z = c_B (B^{-1} b - B^{-1} N x_N) + c_N x_N$$

$$= c_B B^{-1} b + (c_N - c_B B^{-1} N) x_N$$

so that the complete tableau is

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$x_B$</th>
<th>$x_N$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_B$</td>
<td>0</td>
<td>$I$</td>
<td>$B^{-1} N$</td>
<td>$B^{-1} b$</td>
</tr>
<tr>
<td>$-z$</td>
<td>1</td>
<td>0</td>
<td>$c_N - c_B B^{-1} N$</td>
<td>$-c_B B^{-1} b$</td>
</tr>
</tbody>
</table>

We define an important set of values $\bar{y}_1, \ldots, \bar{y}_m$ for this tableau, called the **shadow prices**, by setting

$$\bar{y} = c_B B^{-1}$$

Using this, and writing $B^{-1} = \bar{S} = \begin{pmatrix} \bar{s}_{11} & \cdots & \bar{s}_{1m} \\ \vdots & \ddots & \vdots \\ \bar{s}_{m1} & \cdots & \bar{s}_{mm} \end{pmatrix}$, the tableau can be written

<table>
<thead>
<tr>
<th>basis</th>
<th>$-z$</th>
<th>$x_B$</th>
<th>$x_N$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_B$</td>
<td>0</td>
<td>$I$</td>
<td>$\bar{S} N$</td>
<td>$\bar{S} b$ or simply</td>
</tr>
<tr>
<td>$-z$</td>
<td>1</td>
<td>0</td>
<td>$c_N - \bar{y} N$</td>
<td>$-\bar{y} b$</td>
</tr>
</tbody>
</table>

More on the role of shadow prices later.
**Formulae for a Basic Tableau**

We can now give the formulae for the current tableau in terms of the original tableau and the $\bar{S}$ and $\bar{y}$ values:

1. $\bar{a}_{ij} = \bar{s}_{i1}a_{1j} + \bar{s}_{i2}a_{2j} + \ldots + \bar{s}_{im}a_{mj}$ \hspace{1cm} $i = 1, \ldots, m$
2. $\bar{b}_i = \bar{s}_{i1}b_1 + \bar{s}_{i2}b_2 + \ldots + \bar{s}_{im}b_m$ \hspace{1cm} $i = 1, \ldots, m$
3. $\bar{c}_j = c_j - \bar{y}_1a_{1j} - \bar{y}_2a_{2j} - \ldots - \bar{y}_m a_{mj}$ \hspace{1cm} $j = 1, \ldots, n$
4. $\bar{z}_0 = \bar{y}_1b_1 + \bar{y}_2b_2 + \ldots + \bar{y}_m b_m$

**Example:** Consider the following equality-form minimization problem:

$$\max cx$$

$$Ax = b$$

$$x \geq 0$$

Where $A$, $b$, and $c$ are given below:

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>0</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-5</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>5</td>
</tr>
<tr>
<td>$c$</td>
<td>-2</td>
<td>-3</td>
<td>4</td>
<td>-3</td>
<td>-1</td>
<td>4</td>
<td>-6</td>
<td></td>
</tr>
</tbody>
</table>

Suppose we want to reconstruct the tableau with starting basis $x_B = (x_4, x_6, x_1)$. 
We have
\[ B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \] and \[ c_B = (-3, 4, -2) \]

and so
\[ \bar{S} = B^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix} \]

\[ \bar{y} = c_B \bar{S} = (3/2, 5/2, -9/2) \].

The final tableau has
\[ \bar{A} = \bar{S}A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 3 & 3 & 1 & -1 & 1 & -5 \\ 1 & 3 & -1 & 0 & -1 & 1 & 3 \\ 1 & 2 & 0 & 1 & 2 & 0 & -2 \end{pmatrix} \]

\[ = \begin{pmatrix} 0 & 1 & 2 & 1 & 1 & 0 & -5 \\ 0 & 2 & 1 & 0 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 & 3 \end{pmatrix} \]

\[ \bar{b} = \bar{S}b = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \]
\[ \tilde{c} = c - \bar{y}A = (-2, -3, 4, -3, -1, 4, -6) \]

\[ -(3/2, 5/2, -9/2) \begin{pmatrix} 0 & 3 & 3 & 1 & -1 & 1 & -5 \\ 1 & 3 & -1 & 0 & -1 & 1 & 3 \\ 1 & 2 & 0 & 1 & 2 & 0 & -2 \end{pmatrix} \]

\[ = (0, -6, 2, 0, 12, 0, -15) \]

\[ \tilde{z}_0 = (3/2, 5/2, -9/2) \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = -8 \]

We apply the formulae to a few entries:

\[ \bar{a}_{23} = \bar{s}_{21}a_{13} + \bar{s}_{22}a_{23} + \bar{s}_{23}a_{33} \]
\[ = (1/2)(3) + (1/2)(-1) + (-1/2)(0) = 1 \]

\[ \bar{b}_3 = \bar{s}_{31}b_1 + \bar{s}_{32}b_2 + \bar{s}_{33}b_3 \]
\[ = (-1/2)(3) + (1/2)(4) + (1/2)(5) = 3 \]

\[ \bar{c}_5 = c_5 - \bar{y}_1a_{15} - \bar{y}_2a_{25} - \bar{y}_3a_{35} \]
\[ = -1 - (3/2)(-1) - (5/2)(-1) - (-9/2)(2) = 12 \]

\[ \bar{z}_0 = \bar{y}_1b_1 + \bar{y}_2b_2 + \bar{y}_3b_3 \]
\[ = (3/2)(3) + (5/2)(4) + (-9/2)(5) = -8 \]
The Revised Simplex Method

The Revised Simplex Method is an implementation of the standard Simplex Method that uses an abbreviated tableau, reconstructing only that data from the complete simplex tableaus absolutely necessary to perform the steps of the simplex method.

**Revised tableau:** The only values maintained in the revised tableau are:

- the list of basic variables \( x_B = (x_{B_1}, \ldots, x_{B_m}) \);
- the inverse matrix \( \bar{S} = B^{-1} \), where \( B = [A_{B_1}, \ldots, A_{B_m}] \);
- the right-hand-side values \( \bar{b} = \bar{b}_1, \ldots, \bar{b}_m \),
- the objective function value \( \bar{z}_0 \), and
- the values of \( c_B B^{-1} = \bar{y} = (\bar{y}_1, \ldots, \bar{y}_m) \).

**Tableau appearance:**

<table>
<thead>
<tr>
<th>basis</th>
<th>( \bar{S} ) &amp; ( \bar{y} )</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_{B_1} )</td>
<td>( \bar{s}<em>{11} ) &amp; ( \bar{s}</em>{12} ) &amp; ( \cdots ) &amp; ( \bar{s}_{1m} )</td>
<td>( \bar{b}_1 )</td>
</tr>
<tr>
<td>( x_{B_2} )</td>
<td>( \bar{s}<em>{21} ) &amp; ( \bar{s}</em>{22} ) &amp; ( \cdots ) &amp; ( \bar{s}_{2m} )</td>
<td>( \bar{b}_2 )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots ) &amp; ( \vdots ) &amp; ( \cdots ) &amp; ( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( x_{B_m} )</td>
<td>( \bar{s}<em>{m1} ) &amp; ( \bar{s}</em>{m2} ) &amp; ( \cdots ) &amp; ( \bar{s}_{mm} )</td>
<td>( \bar{b}_m )</td>
</tr>
<tr>
<td>( -z )</td>
<td>( -\bar{y}_1 ) &amp; ( -\bar{y}_2 ) &amp; ( \cdots ) &amp; ( -\bar{y}_m ) &amp; ( -\bar{z}_0 )</td>
<td></td>
</tr>
</tbody>
</table>
The Phase II Revised Simplex Method

As in the standard Phase II Simplex Method, we assume that we start with a feasible basis $x_B$, with the associated values of $\bar{S}$, $\bar{b}$, $\bar{z}$, and $\bar{y}$ known.

Current basic feasible solution:
$x_{Bi} = \bar{b}_i$, $i = 1, \ldots, m$, $x_j = 0$, $x_j$ nonbasic.

Objective function value of current solution: $\bar{z}_0$

Steps of the Revised Simplex Method

Step 1  Find the most negative objective row coefficient: reconstruct the objective row coefficients by setting $\bar{c} = c - \bar{y}A$.
Choose among the $\bar{c}$ the most negative coefficient $\bar{c}_j$. (If none, then the current solution is optimal.)

Step 2  Perform the minimum ratio test on pivot column $j$ to find the leaving variable: reconstruct the pivot column $\bar{A}_j$ by using $\bar{A}_j = \bar{S}A_j$.
Apply the minimum ratio test to this column and the right-hand-side values $\bar{b}$ to find the pivot row $i$ corresponding to leaving variable $x_{Bi}$. (If the reconstructed column is negative, then the LP is unbounded.)
**Step 3** Perform a pivot on row \(i\) and column \(j\): **insert reconstructed column** \(j\) next to the inverse matrix and perform a pivot on the associated row, replacing the blocking variable \(x_{B_i}\) with the entering variable \(x_j\). The updated values of \(\bar{S}, \bar{b}, \bar{z},\) and \(\bar{y}\) are then the correct ones for the new basis.

<table>
<thead>
<tr>
<th>basis</th>
<th>(\bar{S} &amp; - \bar{y})</th>
<th>(x_j)</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_{B_1})</td>
<td>(\bar{s}<em>{11} \ldots \bar{s}</em>{1m})</td>
<td>(\bar{a}_{1j})</td>
<td>(b_1)</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
<tr>
<td>(x_{B_i})</td>
<td>(\bar{s}<em>{i1} \ldots \bar{s}</em>{im})</td>
<td>(\boxed{\bar{a}_{ij}})</td>
<td>(\bar{b}_i)</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
<tr>
<td>(x_{B_m})</td>
<td>(\bar{s}<em>{m1} \ldots \bar{s}</em>{mm})</td>
<td>(\bar{a}_{mj})</td>
<td>(\bar{b}_m)</td>
</tr>
<tr>
<td>(\bar{z})</td>
<td>(\bar{y}_1 \ldots \bar{y}_m)</td>
<td>(\bar{c}_j)</td>
<td>(\bar{z}_0)</td>
</tr>
</tbody>
</table>
Example

Consider the same LP as we had above, with starting basis \( x_B = (x_4, x_6, x_1) \). Using the \( \bar{S}, \bar{b}, \bar{y}, \) and \( \bar{z}_0 \) as computed above, we get starting revised tableau

<table>
<thead>
<tr>
<th>basis</th>
<th>( \bar{S} ) &amp; ( \bar{y} )</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_4 )</td>
<td>1/2 ( -1/2 ) 1/2</td>
<td>2</td>
</tr>
<tr>
<td>( x_6 )</td>
<td>1/2 1/2 ( -1/2 )</td>
<td>1</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>( -1/2 ) 1/2 1/2</td>
<td>3</td>
</tr>
<tr>
<td>( -z )</td>
<td>( -3/2 ) ( -5/2 ) 9/2</td>
<td>8</td>
</tr>
</tbody>
</table>

**Iteration 1:** We have

\[
\bar{c} = (-2, -3, 4, -3, -1, 4, -6) - (3/2, 5/2, -9/2) = (0, -6, -2, 0, 12, 0, -15)
\]

The most negative is the \( \bar{c}_7 = -15 \) term, and so we reconstruct column \( \bar{A}_7 \):

\[
\bar{A}_7 = \begin{pmatrix}
  1/2 & -1/2 & 1/2 \\
  1/2 & 1/2 & -1/2 \\
  -1/2 & 1/2 & 1/2 \\
\end{pmatrix}
\begin{pmatrix}
  -5 \\
  3 \\
  -2 \\
\end{pmatrix}
= \begin{pmatrix}
  -5 \\
  0 \\
  3 \\
\end{pmatrix}
\]

The leaving variable is now \( x_{B_3} = x_1 \), and so we pivot on tableau

<table>
<thead>
<tr>
<th>basis</th>
<th>( \bar{S} ) &amp; ( \bar{y} )</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_4 )</td>
<td>1/2 ( -1/2 ) 1/2</td>
<td>(-5) ( 2 )</td>
</tr>
<tr>
<td>( x_6 )</td>
<td>1/2 1/2 ( -1/2 )</td>
<td>0 ( 1 )</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>( -1/2 ) 1/2 1/2</td>
<td>( 3 ) ( 3 )</td>
</tr>
<tr>
<td>( -z )</td>
<td>( -3/2 ) ( -5/2 ) 9/2</td>
<td>(-15) ( 8 )</td>
</tr>
</tbody>
</table>
to give new revised tableau

\[
\begin{array}{c|cccc|c}
\text{basis} & \bar{S} & -\bar{y} & \text{rhs} \\
\hline
x_4 & -1/3 & 1/3 & 4/3 & 7 \\
x_6 & 1/2 & 1/2 & -1/2 & 1 \\
x_7 & -1/6 & 1/6 & 1/6 & 1 \\
-z & -4 & 0 & 7 & 23 \\
\end{array}
\]

**Iteration 2:** We have

\[
\bar{c} = (-2, -3, 4, -3, -1, 4, -6) - (4, 0, -7) \begin{pmatrix} 0 & 3 & 3 & 1 & -1 & 1 & -5 \\ 1 & 3 & -1 & 0 & -1 & 1 & 3 \\ 1 & 2 & 0 & 1 & 2 & 0 & -2 \end{pmatrix} = (5, -1, -8, 0, 17, 0, 0)
\]

The most negative is the \(\bar{c}_3 = -8\) term, and so we reconstruct the \(\bar{A}_3\) column

\[
\bar{A}_3 = \begin{pmatrix} -1/3 & 1/3 & 4/3 \\ 1/2 & 1/2 & -1/2 \\ -1/6 & 1/6 & 1/6 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4/3 \\ 1 \\ -2/3 \end{pmatrix}
\]

The leaving variable is now \(x_{B_2} = x_6\), and so we pivot on tableau

\[
\begin{array}{c|cccc|c}
\text{basis} & \bar{S} & -\bar{y} & \text{rhs} \\
\hline
x_4 & -1/3 & 1/3 & 4/3 & -4/3 & 7 \\
x_6 & 1/2 & 1/2 & -1/2 & 1 \\
x_7 & -1/6 & 1/6 & 1/6 & -2/3 & 1 \\
-z & -4 & 0 & 7 & -8 & 23 \\
\end{array}
\]
to give new revised tableau

<table>
<thead>
<tr>
<th>basis</th>
<th>$\bar{S}$ &amp; $\bar{y}$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_4$</td>
<td>1/3 1 2/3</td>
<td>25/3</td>
</tr>
<tr>
<td>$x_3$</td>
<td>1/2 1/2 -1/2</td>
<td>1</td>
</tr>
<tr>
<td>$x_7$</td>
<td>1/6 1/2 -1/6</td>
<td>5/3</td>
</tr>
<tr>
<td>$-z$</td>
<td>0 4 3</td>
<td>31</td>
</tr>
</tbody>
</table>

**Iteration 3:** We have

$$\bar{c} = (-2, -3, 4, -3, -1, 4, -6) - (0, -4, -3) \begin{pmatrix} 0 & 3 & 3 & 1 & -1 & 1 & -5 \\ 1 & 3 & -1 & 0 & -1 & 1 & 3 \\ 1 & 2 & 0 & 1 & 2 & 0 & -2 \end{pmatrix}$$

$$= (5, 15, 0, 0, 1, 8, 0)$$

Since all reduced costs are nonnegative, the current solution

$$x_1 = x_2 = 0, x_3 = 1, x_4 = 25/3, x_5 = x_6 = 0, x_7 = 5/3$$

is optimal to the LP, with objective value $z = 31$. 
Comments on Revised Simplex Method

• Requires only $O(m^2)$ storage, as opposed to $O(mn)$ storage for the entire tableau. This means that the pivot step, which is most expensive, also takes only $O(m^2)$ time, rather than $O(mn)$.

• In implementation, the Revised Simplex Method does not keep $B^{-1}$ around at all, but simply keeps the basis matrix $B$ and solves $Bx = b$ to find $\bar{b}$, $xB = c_B$ to find $\bar{y}$, etc. Thus it can take advantage of sparsity of matrices, and use sophisticated techniques like LU decomposition or iterative methods to obtain fast solutions or approximations to the values required to perform a simplex pivot.

• One of the most expensive steps is the generation of the reduced costs. This can be improved by generating a large number of negative reduced cost columns at once, and then performing simplex pivots on these columns, where appropriate, without regenerating the reduced costs before every pivot.

• The Revised Simplex Method is particularly effective for delayed column generation techniques, where the number of columns is so prohibitively large that they must be generated one at a time, rather than all at once at the beginning. More on this later in the course.
**Eta Matrices and Pivoting**

The pivot itself can be performed via a multiplication *on the left* by an *eta* matrix. This matrix is simply a mathematical description of the row operations used to produce the new tableau. Specifically, if we are pivoting on row $i$ in column $j$, and the pivot column is

\[
\begin{pmatrix}
\bar{a}_{1j} \\
\vdots \\
\bar{a}_{mj}
\end{pmatrix},
\]

then the eta matrix for the pivot will be

\[
E = \begin{pmatrix}
1 & 0 & \cdots & -\frac{\bar{a}_{1j}}{\bar{a}_{ij}} & \cdots & 0 \\
0 & 1 & \cdots & -\frac{\bar{a}_{2j}}{\bar{a}_{ij}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{\bar{a}_{ij}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -\frac{\bar{a}_{mj}}{\bar{a}_{ij}} & \cdots & 1
\end{pmatrix}
\]

This allows a quick update of the basis matrix, especially when vector processors are available. If the original matrix inverse is $B_0^{-1}$, and the successive updates have corresponding eta matrices $E_0, \ldots, E_{k-1}$, then the current basis inverse will be

\[
B_k^{-1} = E_{k-1}E_{k-2} \cdots E_1E_0B_0^{-1}
\]
The Phase I Revised Simplex Method

The Phase I Revised Simplex Method is simply the revised form of the Artificial Variable Phase I Method. We start with the artificial Phase I LP:

\[
\begin{align*}
\min w &= x_1^a + \ldots + x_m^a \\
\bar{a}_{11}x_1 + \ldots + \bar{a}_{1n}x_n + x_1^a &= \bar{b}_1 \\
\vdots & \quad \vdots \\
\bar{a}_{m1}x_1 + \ldots + \bar{a}_{mn}x_n + x_m^a &= \bar{b}_m \\
x_1 \geq 0, \ldots, x_n \geq 0, \quad x_1^a \geq 0, \ldots, x_m^a \geq 0
\end{align*}
\]

The initial artificial basis is \( x_B = (x_1^a, \ldots, x_m^a) \), basis matrix \( B = I \), basic costs \( c_B = (1, \ldots, 1) \) and \( \bar{y} \)-values \( c_BB^{-1} = (1, \ldots, 1) \). The starting feasible artificial tableau is now

<table>
<thead>
<tr>
<th>basis</th>
<th>( \bar{S} &amp; -\bar{y} )</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1^a )</td>
<td>1 0 \ldots 0</td>
<td>( b_1 )</td>
</tr>
<tr>
<td>( x_2^a )</td>
<td>0 1 \ldots 0</td>
<td>( b_2 )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( x_m^a )</td>
<td>0 0 \ldots 1</td>
<td>( b_m )</td>
</tr>
<tr>
<td>(-z^a )</td>
<td>(-1 ) (-1 ) \ldots (-1 )</td>
<td>(-\Sigma_i b_i )</td>
</tr>
</tbody>
</table>
We now proceed to apply the Phase II Simplex Method—using the artificial cost vector \((0, \ldots, 0, 1, \ldots, 1)\)—until an optimal solution is reached. If all of the artificial variables are 0, then the current solution is feasible to the original LP. We now only need to replace the artificial \(\bar{y}\) and \(\bar{z}_0\) values with their values for the original LP:

\[
\bar{y} = c_B B^{-1}, \quad \bar{z}_0 = c_B B^{-1}b
\]

to obtain the correct starting revised tableau for the Phase II. We then proceed with the standard Phase II simplex method, using the original objective function for the reduced cost computations (which do not include any of the artificial variables), until either an optimal solution is found or the LP is determined to be unbounded.

**Example:** Consider the problem of finding an initial basic feasible solution to the problem above. The artificial LP for this problem is:

\[
\begin{array}{cccccccccccc}
& x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_1^a & x_2^a & x_3^a & b \\
A^a & 0 & 3 & 3 & 1 & -1 & 1 & -5 & 1 & 0 & 0 & 3 \\
& 1 & 3 & -1 & 0 & -1 & 1 & 3 & 0 & 1 & 0 & 4 \\
& 1 & 2 & 0 & 1 & 2 & 0 & -2 & 0 & 0 & 1 & 5 \\
c^a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{array}
\]

The starting artificial revised tableau for this is

\[
\begin{array}{ccc|c|c|}
\text{basis} & \bar{S} & -\bar{y} & \text{rhs} \\
\hline
x_1^a & 1 & 0 & 0 & 3 \\
x_2^a & 0 & 1 & 0 & 4 \\
x_3^a & 0 & 0 & 1 & 5 \\
-\bar{z}^a & -1 & -1 & -1 & -12 \\
\end{array}
\]
Iteration 1:
\[
\bar{c} = (0, 0, 0, 0, 0, 0, 0, 1, 1, 1) - (1, 1, 1)
\]
\[
\begin{pmatrix}
0 & 3 & 3 & 1 & -1 & 1 & -5 & 1 & 0 & 0 \\
1 & 3 & -1 & 0 & -1 & 1 & 3 & 0 & 1 & 0 \\
1 & 2 & 0 & 1 & 2 & 0 & -2 & 0 & 0 & 1
\end{pmatrix}
\]
\[
= (-2, -8, -2, -2, 0, -2, 4, 0, 0, 0)
\]

\(x_2\) enters, with pivot column
\[
\tilde{A}_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
3 \\
3 \\
2
\end{pmatrix}
= \begin{pmatrix}
3 \\
3 \\
2
\end{pmatrix}
\]

The leaving variable is \(x_1^a\), and the resulting tableau is

<table>
<thead>
<tr>
<th>basis</th>
<th>(\bar{S} &amp; -\bar{y})</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_2)</td>
<td>1/3 0 0</td>
<td>1</td>
</tr>
<tr>
<td>(x_2^a)</td>
<td>-1 1 0</td>
<td>1</td>
</tr>
<tr>
<td>(x_3^a)</td>
<td>-2/3 0 1</td>
<td>3</td>
</tr>
<tr>
<td>(-z^a)</td>
<td>5/3 -1 -1</td>
<td>-4</td>
</tr>
</tbody>
</table>

Iteration 2:
\[
\bar{c} = (0, 0, 0, 0, 0, 0, 0, 1, 1, 1) - (-5/3, 1, 1)
\]
\[
\begin{pmatrix}
0 & 3 & 3 & 1 & -1 & 1 & -5 & 1 & 0 & 0 \\
1 & 3 & -1 & 0 & -1 & 1 & 3 & 0 & 1 & 0 \\
1 & 2 & 0 & 1 & 2 & 0 & -2 & 0 & 0 & 1
\end{pmatrix}
\]
\[
= (-2, 0, 6, 2/3, -8/3, 2/3, -28/3, 8/3, 0, 0)
\]

\(x_7\) enters, with pivot column
\[
\bar{A}_7 = \begin{pmatrix}
1/3 & 0 & 0 \\
-1 & 1 & 0 \\
-2/3 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
-5 \\
3 \\
-2
\end{pmatrix}
= \begin{pmatrix}
-5/3 \\
8 \\
4/3
\end{pmatrix}
The leaving variable is \( x_2 \), and the resulting tableau is

<table>
<thead>
<tr>
<th>basis</th>
<th>( \bar{S} ) &amp; (-\bar{y})</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_2 )</td>
<td>1/8 5/24 0</td>
<td>29/24</td>
</tr>
<tr>
<td>( x_7 )</td>
<td>-1/8 -1/8 0</td>
<td>1/8</td>
</tr>
<tr>
<td>( x_3^a )</td>
<td>-1/2 1/6 1</td>
<td>17/6</td>
</tr>
<tr>
<td>(-z^a)</td>
<td>1/2 1/6 -1</td>
<td>-17/6</td>
</tr>
</tbody>
</table>

**Iteration 3:**

\[
\bar{c} = (0,0,0,0,0,0,1,1,1) - (-1/2,-1/6,1) \begin{pmatrix}
0 & 3 & 3 & 1 & -1 & 1 & -5 & 1 & 0 & 0 \\
1 & 3 & 1 & 0 & -1 & 1 & 3 & 0 & 1 & 0 \\
1 & 2 & 0 & 1 & 2 & 0 & -2 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = (-5/6, 0, 4/3, -1/2, -8/3, 2/3, 0, 3/2, 7/6, 0)
\]

\( x_5 \) enters, with pivot column

\[
\bar{A}_5 = \begin{pmatrix}
1/8 & 5/24 & 0 \\
-1/8 & -1/8 & 0 \\
-1/2 & 1/6 & 1
\end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 0 \\ 8/3 \end{pmatrix}
\]

The leaving variable is \( x_3^a \), and the resulting tableau is

<table>
<thead>
<tr>
<th>basis</th>
<th>( \bar{S} ) &amp; (-\bar{y})</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_2 )</td>
<td>1/16 3/16 1/8</td>
<td>25/16</td>
</tr>
<tr>
<td>( x_7 )</td>
<td>-1/8 1/8 0</td>
<td>1/8</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>-3/16 -1/16 3/8</td>
<td>17/6</td>
</tr>
<tr>
<td>(-z^a)</td>
<td>0 0 0</td>
<td>0</td>
</tr>
</tbody>
</table>
Since the artificial objective is 0 we can begin the Phase II portion of the revised method, by simply computing the initial $\bar{y}$ and $\bar{z}_0$ values as

$$
\bar{y} = (-3, -6, -1) \begin{pmatrix}
    1/16 & 3/16 & 1/8 \\
    -1/8 & 1/8 & 0 \\
    -3/16 & -1/16 & 3/8 \\
\end{pmatrix} = (3/4, -5/4, -3/4)
$$

$$
\bar{z}_0 = (3/4, -5/4, -3/4) \begin{pmatrix}
    3 \\
    4 \\
    5 \\
\end{pmatrix} = -13/2
$$

The starting Phase II tableau is now

<table>
<thead>
<tr>
<th>basis</th>
<th>$\bar{S} &amp; -\bar{y}$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>1/16 3/16 1/8</td>
<td>25/16</td>
</tr>
<tr>
<td>$x_7$</td>
<td>-1/8 1/8 0</td>
<td>1/8</td>
</tr>
<tr>
<td>$x_5$</td>
<td>-3/16 -1/16 3/8</td>
<td>17/6</td>
</tr>
<tr>
<td>$-z$</td>
<td>-3/4 5/4 3/4</td>
<td>13/2</td>
</tr>
</tbody>
</table>

and we continue on with the Phase II Method, using original cost vector $(-2, -3, 4, -3, -1, 4, -6)$. So, for example, the next set of reduced costs will be

$$
\bar{c} = (-2, -3, 4, -3, -1, 4, -6) - (3/4, -5/4, -3/4) \begin{pmatrix}
    0  & 3  & 3 & 1 & -1 & 1 & -5 \\
    1  & 3  & -1 & 0 & -1 & 1 & 3 \\
    1  & 2  & 0  & 1 & 2 & 0 & -2 \\
\end{pmatrix} = (0, 0, 1/2, -3, 0, 9/2, 0).
$$