Linear programming

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Handouts of the *Linear Programming* for *Engineering* students

LECTURE 1
1 Linear Algebra (A Reminder)

Let \( a_i \in \mathbb{R}^m \) and \( i = 1, 2, \ldots, n \). Denote the set of indices belonging to vectors \( a_i \) with \( J \).

**Definition 1** A linear combination of the vectors \( \{ a_j \mid j \in J \} \) is any vector \( b \in \mathbb{R}^m \) of the form

\[
    b = \sum_{j \in J} k_j a_j
\]

where \( k_j \in \mathbb{R} \), for all \( j \in J \).

**Example 2** Let us suppose that

\[
    a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix},
\]

vectors are given. Then \( b \) is a linear combination of \( a_1, a_2, a_3 \) since

\[
    b = -4a_1 + 7a_2 - a_3.
\]

Therefore \( k_1 = -4, k_2 = 7 \) and \( k_3 = -1 \).

**Definition 3** Let \( J_G \subset J \). The set \( \{ a_i \mid i \in J_G \} \) is called a generating system of \( \{ a_j \mid j \in J \} \) if any \( a_j, j \in \bar{J}_G \) (where \( \bar{J}_G = J \setminus J_G \) ) can be expressed as a linear combination of vectors \( \{ a_i \mid i \in J_G \} \), namely the set \( \{ a_j \mid j \in J \} \) is spanned by vectors \( a_i, i \in J_G \).

Let us suppose that \( J_G \subset J \) containing the indices of a generating system. Then

\[
    a_j = \sum_{i \in J_G} t_{ij} a_i \quad \forall j \in \bar{J}_G \quad (1)
\]

From the expression (1) we can produce the following (pivot) tableau

\[
    \begin{array}{cccc}
        \vdots & \vdots & \vdots & \vdots \\
        a_j & \cdots & t_{ij} & \cdots \\
        \vdots & \vdots & \vdots & \vdots \\
    \end{array}
\]

Figure 1
Example 4 Let \( a_1, a_2, a_3 \) and \( b \) be the same vectors as in Example 2. Furthermore, let \( a_4 = b \) and

\[
\begin{pmatrix}
4 \\
1 \\
-1
\end{pmatrix}
\]

then

\[
\begin{array}{ccc}
a_4 & a_5 \\
-4 & -1 \\
7 & 2 \\
-1 & 3
\end{array}
\]

forms the (pivot) tableau.

The \( t_{ij} \neq 0 \) element is called pivot element, the row \( i \in J_G \) is called pivot row, while column \( j \in J_G \) is called pivot column. A pivot operation or a pivot step is uniquely defined by the pivot element. If \( t_{ij} \) is the pivot element then the pivot operation consists of

- a series of elementary row operations on (pivot) tableau, to transform the pivot column into one containing an entry of 1 in the pivot row \( (i \in J_G) \), and an entry 0 in all the other rows,

- exchange the unity vector obtained in the pivot column \( (j \in J_G) \) by \(-t_{kj}/t_{ij}\) when \( k \neq i \) and with \( 1/t_{ij} \) when \( k = j \),

\( J_G := (J_G \setminus \{i\}) \cup \{j\} \) and \( \tilde{J}_G := (\tilde{J}_G \setminus \{j\}) \cup \{i\} \).

The result of pivot operation can be summarized in the following theorem.

Theorem 5 If \( t_{rs} \neq 0 \) then \( a_r, r \in J_G \) can be exchanged with \( a_s, s \in \tilde{J}_G \) in the following way:

\[
\begin{align*}
(1) & \quad t'_{ij} = t_{ij} - \frac{t_{ri}t_{is}}{t_{rs}} \quad i \in J'_G, \quad i \neq s, \quad j \in \tilde{J}_G', \quad j \neq r \\
(2) & \quad t'_{sj} = \frac{t_{rs}}{t_{rs}} \quad \quad j \in \tilde{J}_G', \quad j \neq r \\
(3) & \quad t'_{ir} = -\frac{t_{is}}{t_{rs}} \quad i \in J'_G, \quad i \neq s \\
(4) & \quad t'_{sr} = \frac{1}{t_{rs}}
\end{align*}
\]

where \( J'_G = (J_G \setminus \{r\}) \cup \{s\} \) is the index of the new generating system and the scalars are \( t'_{ij} \) in the new expressions.

Proof: The pivot tableau corresponding to the generating system given by its index set \( J_G \) is given below
Make a pivot on \((r, s)\) position, \(t_{rs} \neq 0\). Then we get the following tableau:

Let us start from the expression of \(a_s, s \in \bar{J}_G\), thus

\[
a_s = t_{rs}a_r + \sum_{i \in J_G \setminus \{r\}} t_{is}a_i
\]

then

\[
a_r = \frac{1}{t_{rs}}a_s + \sum_{i \in J_G \setminus \{r\}} \left(\frac{-t_{is}}{t_{rs}}\right)a_i
\]

So, we obtained expressions (3) and (4). The vector \(a_j, j \in \bar{J}_G, j \neq s\) is a linear combination of those from the generating system,

\[
a_j = t_{rj}a_r + \sum_{i \in J_G \setminus \{r\}} t_{ij}a_i.
\]

When we substitute vector \(a_r\) into the previous expression we obtain
\[
\mathbf{a}_j = \frac{t_{rj}}{t_{rs}} \mathbf{a}_s + \sum_{i \in \bar{J}_G \setminus \{r\}} \left( t_{ij} - \frac{t_{rj}t_{is}}{t_{rs}} \right) \mathbf{a}_i
\]

where \( j \in \bar{J}_G, j \neq r \). Thus (1) and (2) is proved. \( \square \)

**Example 6** Let \( \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \) and \( \mathbf{a}_5 \) be the same vectors as in Example 4 and the corresponding (pivot) tableau, as well.

We may choose \( t_{34} = -1 \neq 0 \) as a pivot element. Applying Theorem 5 with \( r = 3, s = 4 \) the new pivot tableau is

\[
\begin{array}{ccc}
\mathbf{a}_3 & \mathbf{a}_5 \\
\mathbf{a}_1 & -4 & -13 \\
\mathbf{a}_2 & 7 & 23 \\
\mathbf{a}_4 & -1 & -3 \\
\end{array}
\]

Using matrix representation we can introduce the following tableau, which describes the relationship clearly and briefly between vectors of generating system and those not belonging to the generating system. The row are allocated to the generator vectors, while the columns are splitted into two different groups. First group contains vectors not belonging to the generating system and the second group is formed by generator vectors. Then

\[
\begin{array}{cc}
\bar{J}_G & J_G \\
J_G & \begin{bmatrix} T & I \end{bmatrix} \\
\end{array}
\]

Figure 4

where \( T = (t_{ij}), i \in J_G, j \in \bar{J}_G \) and the unity matrix \( I \) means that any vector \( \mathbf{a}_i, i \in J_G \) can be expressed using vectors from the set \( \{ \mathbf{a}_j | j \in \bar{J}_G \} \) in trivial way, i.e

\[
\mathbf{a}_i = \mathbf{1} \mathbf{a}_i + \sum_{j \in J_G \setminus \{i\}} 0 \mathbf{a}_j.
\]

The previous tableau we shall call it, pivot tableau or complete tableau, on the other hand the following tableau we shall call short tableau or dictionary.

\[
\begin{array}{c}
\bar{J}_G \\
J_G & \begin{bmatrix} T \end{bmatrix} \\
\end{array}
\]

Figure 5
Definition 7 The set \( \{ a_j \in \mathbb{R}^m | j \in J \} \) of vectors, where \( |J| \geq 2 \), is said to be linearly independent, if \( \nexists \ a_r, r \in J \) vector which can be expressed as a linear combination of the vectors \( \{ a_j \in \mathbb{R}^m | j \in J \setminus \{ r \} \} \).

The set of vectors \( \{ a_1, a_2, a_3, a_4 \} \) is not linearly independent (see Example 2 and 4).

Exercise 8

1. Let us suppose that \( \{ a_j \in \mathbb{R}^m | j \in J \} \) is a system of linearly independent vectors. Decide, whether the following statements are true or not. Give the explanations, too.

   (i) \( 0 \in \{ a_j \in \mathbb{R}^m | j \in J \} \)
   (ii) \( |J| = 1 \)

2. Show that any subset of linearly independent vectors is linearly independent set of vectors.

3. Let the set of vectors \( \{ a_1, a_2, \ldots, a_n \} \subset \mathbb{R}^m \) be linearly independent and for a given \( b \in \mathbb{R}^m, b \neq 0 \),

\[
b = \sum_{i=1}^{n} k_i a_i
\]

then if \( k_i \neq 0 \) we may exchange vectors \( a_i \) and \( b \). Show that the following set of vectors \( \{ a_1, a_2, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n \} \) is linearly independent.

4. The system \( \{ a_j \in \mathbb{R}^m | j \in J \} \) of vectors is given. Let us suppose that \( |J| \geq 2 \). Then the following statements are equivalent:

   (i) the system of vectors is linearly independent,
   (ii) if \( b \in \mathbb{R}^m \) can be expressed as a linear combination of vectors from \( \{ a_j \in \mathbb{R}^m | j \in J \} \), then it is possible only in a unique way,
   (iii) if \( \sum_{i \in J} k_j a_j = 0 \), then \( k_j = 0 \ \forall j \in J \)

The system \( \{ a_j | j \in J \}, |J| = 1 \) is said to be linearly independent if \( a_1 \neq 0 \).

Let \( e_i \in \mathbb{R}^m \) denotes the \( i^{th} \) unit vector then for any vector \( b \in \mathbb{R}^m \)

\[
b = b_1 e_1 + b_2 e_2 + \ldots + b_m e_m
\]

therefore \( \{ e_1, e_2, \ldots, e_m \} \) forms a generating system of any set of vectors\(^1\) \( \{ a_1, a_2, \ldots, a_n \} \subset \mathbb{R}^m \), as well. The corresponding (short) pivot tableau is

\(^1\)We may simply say that \( \{ e_1, e_2, \ldots, e_m \} \) span \( \mathbb{R}^m \).
In this case \( T = A \) where \( A \) is an \( m \times n \) real matrix, whose column vectors are \( a_1, a_2, \ldots, a_n \).

**Algorithm 9 (for testing linear independence)**

Let the set of vectors \( \{a_1, a_2, \ldots, a_n\} \subset \mathbb{R}^m \) be given. Arrange them as in Figure 6. Let us denote the index set of the vectors \( \{a_1, a_2, \ldots, a_n\} \) by \( \mathcal{I} \) and the index set of the unity vectors by \( \mathcal{J} \). Furthermore, let \( \mathcal{J}_G := \mathcal{I} \) and \( \bar{\mathcal{J}}_G := \mathcal{J} \).

**Step 1** Choose any \( j \in \mathcal{J} \cap \bar{\mathcal{J}}_G \). If for all \( i \in \mathcal{I} \cap \mathcal{J}_G : t_{ij} = 0 \)
then STOP, vectors \( a_1, a_2, \ldots, a_n \) are linearly dependent
otherwise choose any \( i \in \mathcal{I} \cap \mathcal{J}_G : t_{ij} \neq 0 \), pivot at \((i, j)\) position,
\( \mathcal{J}_G := (\mathcal{J}_G \cup \{j\}) \setminus \{i\} \).
Go to Step 2.

**Step 2** If \( \mathcal{J} \cap \bar{\mathcal{J}}_G = \emptyset \)
then STOP, vectors \( a_1, a_2, \ldots, a_n \) are linearly independent
otherwise if \( \mathcal{I} \cap \mathcal{J}_G = \emptyset \)
then STOP, vectors \( a_1, a_2, \ldots, a_n \) are linearly dependent;
otherwise go to Step 1.

The algorithm checks every column of the pivot tableau at most once, therefore the algorithm stops after at most \( n \) iterations.

**Example 10** Check whether the set of column vectors of the matrix \( A \) is a linearly independent set, where

\[
A = \begin{pmatrix}
0 & 1 & 1 & 2 \\
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
0 & 1 & 1 & 2 \\
1 & 0 & 1 & 2
\end{pmatrix}.
\]

Let us form a pivot tableau from the column vectors \( a_1, a_2, a_3, a_4 \) and the unity vectors \( e_1, e_2, e_3, e_4, e_5 \).
Now \( J \cap \tilde{J}_G = \{a_1, a_2, a_3, a_4\} \) and \( I \cap J_G = \{e_1, e_2, e_3, e_4, e_5\} \). We may exchange vectors \( e_2 \) and \( a_1 \).

After the previous pivot we have \( J \cap \tilde{J}_G = \{a_2, a_3, a_4\} \) and \( I \cap J_G = \{e_1, e_3, e_4, e_5\} \). Now a possible pivot position is \( t_{12} \).

At current tableau we have \( J \cap \tilde{J}_G = \{a_3, a_4\} \) and \( I \cap J_G = \{e_3, e_4, e_5\} \). A possible pivot position is \( t_{33} \).

Finally we get \( J \cap \tilde{J}_G = \{a_4\} \) and \( I \cap J_G = \{e_4, e_5\} \). But \( t_{44} = 0 \) and \( t_{54} = 0 \) therefore \( a_1, a_2, a_3, a_4 \) are linearly dependent vectors (the algorithm stops at Step 1) and

\[ a_4 = a_1 + a_2 + a_3. \]

**Theorem 11 (Steinitz):** Let the system of vectors \( \{a_j \in R^n | j \in \mathcal{J}\} \) be given and \( \mathcal{J}_F \subseteq \mathcal{J} \) and \( \mathcal{J}_G \subseteq \mathcal{J} \) denote the index sets of the linearly independent vectors and generator vectors, respectively. Then \( |\mathcal{J}_F| \leq |\mathcal{J}_G| \) holds.
**Proof:** Without loss of generality we can assume that \( \mathcal{J} = \mathcal{J}_G \cup \mathcal{J}_F \) (because if we add some vectors to the generating system, it remains a generating system, as well). Suppose on contrary that \( |\mathcal{J}_F| > |\mathcal{J}_G| \).

Consider the expression of the vectors \( \{a_j \in R^m | j \in \mathcal{J}_G\} \) using generator vectors. This can be shown in the following way using the dictionary (short tableau):

\[
\begin{array}{ccc}
\mathcal{J}_G \cap \mathcal{J}_F & \mathcal{J}_G \cap \tilde{\mathcal{J}}_F \\
\mathcal{J}_G \cap \mathcal{J}_F & a_i & \cdots & t_{ij} & \cdots \\
& \vdots & \vdots \\
\end{array}
\]

Figure 7

Make pivot at \( t_{ij} \neq 0 \) position, if \( i \in \mathcal{J}_G \cap \mathcal{J}_F \) and \( j \in \tilde{\mathcal{J}}_G \cap \mathcal{J}_F \). Repeat such a pivot until it is possible. Then the following tableau is obtained

\[
\begin{array}{ccc}
\mathcal{J}_F \cap \tilde{\mathcal{J}}_G & \mathcal{J}_F \cap \mathcal{J}_G' \\
\mathcal{J}_G' \cap \mathcal{J}_F & 0 \\
\mathcal{J}_G' \cap \mathcal{J}_F & \mathcal{J}_G' \cap \mathcal{J}_F \\
\end{array}
\]

Figure 8

For all \( i \in \mathcal{J}_G' \cap \mathcal{J}_F \) and \( j \in \mathcal{J}_F \cap \mathcal{J}_G' \), \( t_{ij} = 0 \) because the pivoting procedure stopped. Then

(a) \( \tilde{\mathcal{J}}_G \cap \mathcal{J}_F \neq \emptyset \)

because, if \( \tilde{\mathcal{J}}_G \cap \mathcal{J}_F = \emptyset \), then every linearly independent vector has to be in \( \{a_j | j \in \mathcal{J}_G\} \), means that \( |\mathcal{J}_F| \leq |\mathcal{J}_G| \) but this contradicts to our assumption.

(b) \( \mathcal{J}_G' \cap \mathcal{J}_F \neq \emptyset \),

because, if \( \mathcal{J}_G' \cap \mathcal{J}_F = \emptyset \) then we get the following tableau:

\[
\begin{array}{ccc}
\mathcal{J}_F \cap \mathcal{J}_G' & \mathcal{J}_F \cap \mathcal{J}_G' \\
\mathcal{J}_G' \cap \mathcal{J}_F & 0 \\
\mathcal{J}_G' \cap \mathcal{J}_F & \mathcal{J}_G' \cap \mathcal{J}_F \\
\end{array}
\]
Then, from (a), we have
\[ \sum_{i \in \mathcal{J}_G' \cap \mathcal{J}_F} 0a_i = 0 \]

This means that \( 0 \in \{ a_j | j \in \mathcal{J}_F \} \) which contradicts to the fact that \( \{ a_j | j \in \mathcal{J}_F \} \) is a linearly independent system of vectors. Summarizing the statements (a) and (b) we obtain that the vectors belonging to the index set \( \mathcal{J}_G' \cap \mathcal{J}_F \) can be expressed with vectors belonging to the index set \( \mathcal{J}_G' \cap \mathcal{J}_F \), which means there is (at least) one vector from the index set \( \mathcal{J}_F \) which is a linear combination of some other vectors from the same index set. This contradicts to the fact that \( \mathcal{J}_F \) is the index set of linearly independent vectors. This completes our proof. \( \square \)

**Definition 12** If the system of vectors \( \{ a_i | i \in \mathcal{J}_B \} \), \( \mathcal{J}_B \subseteq \mathcal{J} \) is a linearly independent and generating system of \( \{ a_j | j \in \mathcal{J} \} \) then it is called a basis.

**Example 13** Let \( a_1, a_2, a_3, a_4 \) be the same vectors as in Example 10 then \( a_1, a_2, a_3 \) are linearly independent and span \( \{ a_1, a_2, a_3, a_4 \} \) because \( a_4 = a_1 + a_2 + a_3 \). Therefore \( a_1, a_2, a_3 \) forms a basis of the set of vectors \( \{ a_1, a_2, a_3, a_4 \} \).

**Exercise 14** (Basis Theorem)
If \( \mathcal{J}_B', \mathcal{J}_B'' \subset \mathcal{J} \) are the index sets of two different basis of \( \{ a_j | j \in \mathcal{J} \} \). Then
\[ |\mathcal{J}_B'| = |\mathcal{J}_B''| \]

**Definition 15** Let \( \{ a_1, a_2, \cdots, a_n \} \) be a given set of vectors. The rank of the given set of vectors equal to the number of elements of the basis of \( \{ a_j | j \in \mathcal{J} \} \), thus \( \text{rank}(a_1, a_2, \cdots, a_n) = |\mathcal{J}_B| \), where \( \mathcal{J}_B \subset \mathcal{J} \) the index set of an arbitrary basis.

**Example 16** The \( \text{rank}(a_1, a_2, a_3, a_4) = 3 \) because the basis contains three vectors out of four (see Example 10 and 13).

**Exercise 17** \( \mathcal{J}_B, \mathcal{J}_B' \subset \mathcal{J} \) are the index set of two different basis of \( \{ a_j | j \in \mathcal{J} \} \) then from the basis \( \mathcal{J}_B' \) with one pivot we can move to a basis \( \mathcal{J}_B'' \) such as
\[ |\mathcal{J}_B \cap \mathcal{J}_B'| = |\mathcal{J}_B' \cap \mathcal{J}_B''| - 1. \]

**Definition 18** The following set
\[ L(a_1, a_2, \cdots, a_n) = \{ b \in \mathbb{R}^m | b = \sum_{i=1}^{n} x_i a_i, \quad \forall x_i \in \mathbb{R}, \quad i = 1, 2, \cdots, n \} \]
is called a subspace generated by vectors \( a_1, a_2, \cdots, a_n \).
Example 19 Let $a_1, a_2, a_3, a_4, a_5$ be the same vectors as in Example 4. Then

$L(a_1, a_2, a_3, a_4, a_5) = \{ b \in \mathbb{R}^3 \mid b = x_1a_1 + x_2a_2 + x_3a_3 + x_4a_4 + x_5a_5, \ x_1, x_2, x_3, x_4, x_5 \in \mathbb{R} \}$

is a linear subspace of $\mathbb{R}^3$ generated by $a_1, a_2, a_3, a_4, a_5$.

The equation above can be expressed as

\begin{align*}
    b_1 &= x_1 + x_2 + x_3 + 2x_4 + 4x_5 \\
    b_2 &= x_1 + x_2 + 3x_4 + x_5 \\
    b_3 &= x_1 - 4x_4 - x_5
\end{align*}

Exercise 20 1. Prove the following statements:

(i) $L(a_1, \ldots, a_j, \ldots, a_n) = L(a_1, \ldots, \lambda a_j, \ldots, a_n)$ where $\lambda \in \mathbb{R}, \lambda \neq 0$ is given.

(ii) $L(a_1, \ldots, a_j, \ldots, a_k, \ldots, a_n) = L(a_1, \ldots, a_j + a_k, \ldots, a_k, \ldots, a_n), \ 1 \leq k \leq n$.

(iii) $\text{rank}(a_1, a_2, \ldots, a_n) = \text{rank} \ L(a_1, a_2, \ldots, a_n)$.

2. Let $J_{B_1}, J_{B_2} \subset J$ be the index sets of two different basis of $\{a_j \mid j \in J \}$ and $\hat{J} \subset J$ an arbitrary index set of that, then

$L(\hat{t}_1^{(i)} | i \in J_{B_1}) = L(\hat{t}_2^{(i)} | i \in J_{B_2})$,

where $\hat{t}_1^{(i)}$ and $\hat{t}_2^{(i)}$ are the $i^{th}$ row vectors of the following tableau’s

\[ \text{Figure 10} \]

Theorem 21 (Matrix Rank Theorem) Let $A \in \mathbb{R}^{m \times n}$ arbitrary matrix. $A = (a_1, a_2, \ldots, a_n)$ where $a_j \in \mathbb{R}^m, \ j = 1, 2, \ldots, n$ are column vectors and $A = (a^{(1)}, a^{(2)}, \ldots, a^{(m)})$ where $a^{(i)} \in \mathbb{R}^n, \ i = 1, 2, \ldots, m$ are row vectors of the matrix $A$. Then

$\text{rank}(a_1, a_2, \ldots, a_n) = \text{rank}(a^{(1)}, a^{(2)}, \ldots, a^{(m)})$. 
Proof: The matrix $A$ can be represented as the dictionary of the following vectors \{${a_1, a_2, ..., a_n, e_1, e_2, ..., e_m}$\} where \{${e_1, e_2, ..., e_m}$\} form a basis of the previous set of vectors.\(^2\) Using pivot, exchange as many $e_i$ vectors with $a_j$ vectors as possible. Then we obtain the following tableau.

\[
\begin{array}{cccc}
J_B & \hat{J}_B & I_B & I_B \\
1 & 1 & \hat{t}^{(k)} & 0 \\
1 & & & \\
0 & 0 & 1 & \ddots \\
\end{array}
\]

Figure 11

where $J_B \subset J$ is the index set of those vectors from the \{${a_j | j \in J}$\} which become basic vectors, and $I_B$ is the index set of those unity vectors which still remains basic vectors.

Applying the result of the Exercise 1.4.2, we get the following

\[ L(a^{(1)}, ..., a^{(m)}) = L(\hat{t}^{(k)} | k \in J_B), \]

because those row vectors which belong to $I_B$ and $J$ are zero vectors (since the pivoting stopped). By the statement (iii) of Exercise 1.4.1 and using the previous remark we get

\[ rank (a^{(1)}, a^{(2)}, ..., a^{(m)}) = rank L(a^{(1)}, a^{(2)}, ..., a^{(m)}) = rank L(\hat{t}^{(k)} | k \in J_B) = |J_B| \]

The last equality holds because the row vectors $\hat{t}^{(k)}$ are linearly independent vectors. That part of the matrix contains a unity matrix as a submatrix. From the other hand $\{a_j | j \in J_B\}$ form a basis of $\{a_1, a_2, ..., a_n\}$ and then by the definition of the rank, follows that

\[ rank(a_1, a_2, ..., a_n) = |J_B| \]

This completes our proof.\(^\square\)

Example 22 Let $a_1, a_2, a_3, a_4, a_5$ be column vectors of the matrix $A$ of size 3 by 5, as in Example 14. This matrix can be represented as the dictionary part of the following pivot tableau\(^3\)

\(^2\)The $e_i = (0, ..., 0, 1, 0, ..., 0)$ is known as the $i^{th}$ unity vector of appropriate length. In our case the length is $m$.

\(^3\)Let us denote the vectors placed as rows in matrix $A$ by $a^{(1)}, a^{(2)}$ and $a^{(3)}$, respectively.
We would like to illustrate the matrix rank theorem, therefore using pivot, exchange as many \( e_i \) vectors with \( a_j \) vectors as possible.

First we may exchange vectors \( a_3 \) and \( e_1 \). After this trivial step we may exchange vectors \( a_2 \) and \( e_2 \).

During the next pivot step vectors \( a_1 \) and \( e_3 \) are exchanged.

Let \( \hat{a}^{(1)} = (0, 0, 1, -1, 3) \) be defined by the first five entries of the first row of the last tableau. Vectors \( \hat{a}^{(2)} \) and \( \hat{a}^{(3)} \) can be defined similarly, using second and third rows, respectively.

Now \( L(a^{(1)}, a^{(2)}, a^{(3)}) = L(\hat{a}^{(1)}, \hat{a}^{(2)}, \hat{a}^{(3)}) \) because the third tableau is obtained from the original one by applying row elementary operations, therefore \( \text{rank}(a^{(1)}, a^{(2)}, a^{(3)}) = 3 \).

From the other hand \( \{a_1, a_2, a_3\} \) forms a basis of \( \{a_1, a_2, a_3, a_4, a_5\} \) therefore

\[
\text{rank}(a_1, a_2, a_3, a_4, a_5) = 3
\]

\[\text{Definition 23}\]

Let \( a, b \in \mathbb{R}^n \) given vectors. The \( a \) and \( b \) vectors is said to be orthogonal vectors, if \( a^T b = 0 \).

\[\text{Exercise 24}\]

Let \( y \in \mathbb{R}^m \) such a vector that \( y^T a_i = 0, i = 1, 2, \ldots, n \). Then for all \( b \in L(a_1, a_2, \ldots, a_n) : y^T b = 0 \).

Using the pivot tableau, let us introduce the following n-tuples.

\[ t^{(i)} = \left( t^{(i)}_k \right)_{k=1}^{n} = \begin{cases} 
\tilde{t}_{ik}, & \text{if } k \in \bar{J}_B \\
1, & \text{if } k = i \\
0, & \text{if } k \in J_B, k \neq i 
\end{cases} \quad i \in J_B \]
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\[
t^{(j)} = \left( t^{(j)}_k \right)_{k=1}^n = \begin{cases} 
t_{kj}, & \text{if } k \in \mathcal{J}_B \\
1, & \text{if } k = j \\
0, & \text{if } k \in \mathcal{J}_B, \ k \neq j \end{cases}
\]

where \( t^{(i)}, i \in \mathcal{J}_B \) is a column vector defined by the \( i \)th row of the pivot tableau corresponding to the basis \( \mathcal{J}_B \), while \( t_j, j \in \mathcal{J}_B \) is a column vector defined by the \( j \)th column of the dictionary extended with an \((n - m)\) length negative unity vector.

**Example 25** Using the pivot tableau given in the Example 4 and the fact that \( a_1, a_2, a_3 \) are linearly independent vectors which generates \( a_4 \) and \( a_5 \), as well. We can read out the following (column) vectors

\[
t^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -4 \\ -1 \end{pmatrix}, \quad t^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 7 \\ 2 \end{pmatrix}, \quad t^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 3 \end{pmatrix}, \quad t_4 = \begin{pmatrix} -4 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad t_5 = \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix}
\]

It can be easily verified that

\[ t^T_j t^{(i)} = 0 \]

where \( i = 1, 2, 3 \) and \( j = 4, 5 \). Therefore \( t_4 \) and \( t_5 \) are orthogonal to \( t^{(1)}, t^{(2)} \) and \( t^{(3)} \).

**Theorem 26** (Orthogonality Theorem) Let \( \{a_j \mid j \in \mathcal{J}\} \subset \mathbb{R}^m \) be an arbitrary system of vectors and \( B', B'' \) are its basises. Then

\[ t^{(i)}_j t''_j = 0 \quad \text{for all } i \in \mathcal{J}_{B'} \quad \text{and for all } j \in \bar{\mathcal{J}}_{B''}, \]

where \( \mathcal{J}_{B'} \) and \( \mathcal{J}_{B''} \) are the index sets of basises \( B' \) and \( B'' \), respectively.

**Proof** : First we will show that this property holds if \( B' = B'' \). Thus we need to prove the following statement

\[ t''_j t''_j = 0 \quad \text{for all } i \in \mathcal{J}_{B''} \quad \text{and for all } j \in \bar{\mathcal{J}}_{B''}, \]

hence the structures of \( t''_j \) and \( t''_j \) are the following

\[
t''_j = \begin{array}{ccccccc}
* & \cdots & * & t''_{ij} & * & \cdots & *
\end{array}
\]

\[
\begin{array}{ccccccc}
 J_{B''} & & \bar{J}_{B''} \\
0 & \cdots & 0 & 1 & \cdots & 0 & * & \cdots & * & t''_{ij} & * & \cdots & *
\end{array}
\]

Figure 12
thus
\[ t^{n(i)T}t_j^{''} = 1t^{''}_{ij} + (-1)t^{''}_{ij} = 0. \]

Which means that \( t^{''}_j \) is orthogonal to the row-space of the basic tableau generated by \( B^{''} \).

But the row-spaces generated by different basic tableau’s are just the same (statement of Exercise 1.4.2) and the different basic tableaus can be transfered to each other by a sequence of pivot transformations (statement of Exercise 1.3). This completes the proof. \( \Box \)

**Corollary 27 (Composition Property)** Let \( a_1, a_2, \ldots, a_n, e_1, e_2, \ldots, e_m \in \mathbb{R}^m \) vectors be given and \( J_B \subset J = \{1, 2, \ldots, n\}, I_B \subset \{1, 2, \ldots, m\} \) index sets. The \( J_B \cup I_B \) index set is the index set of the basis. Then \( t_{kj} = y^{(k)T}a_j \), where \( k \in J_B \cup I_B \) and \( j \in J \).

![Figure 13](image1.png)

**Proof:** Apply the orthogonality theorem to the \( t_j \) column vector corresponding to the first pivot tableau and to the row vector \( t^{(k)} \) of the second pivot tableau.

Thus we have
\[ 0 = t_j^Tt^{(k)} = -1t_{kj} + a_j^Ty^{(k)} \]
from which follows \( t_{kj} = y^{(k)T}a_j \). \( \Box \)