1. 3 fair coins are tossed. Let $X$ denote the number of heads, and $Y$ denote the 'number of heads' minus 'number of tails'. Find the joint probability distribution of $X$ and $Y$, with marginal distributions.

2. A fair coin is tossed 3 times. Let $X$ denote number of heads, and $Y$ denote

\[ Y = \begin{cases} 0, & \text{if first toss is tail} \\ 1, & \text{if first toss is head} \end{cases} \]

Determine the joint distribution of $X$ and $Y$, with marginal distributions.

3. A pair of dice is tossed. Let $X(a,b) = \max(a,b)$ and $Y(a,b) = a + b$. Determine the joint distribution of $X$ and $Y$, with marginal distributions.

4. Suppose that $X$ and $Y$ have the following discrete joint probability distribution.

<table>
<thead>
<tr>
<th>$Y(S)$</th>
<th>$X(S)$</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.10</td>
<td>0.15</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.20</td>
<td>0.30</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.10</td>
<td>0.15</td>
<td></td>
</tr>
</tbody>
</table>

Find the
a) marginal distributions for $X$ and $Y$.
b) joint expectation of $X$ and $Y$.
c) covariance of $X$ and $Y$.
d) correlation of $X$ and $Y$, and identify it.
e) expectation of $E(2X - 3Y + 1)$.

5. Given the following discrete joint distribution of $X$ and $Y$.

<table>
<thead>
<tr>
<th>$Y(S)$</th>
<th>$X(S)$</th>
<th>-4</th>
<th>2</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/8</td>
<td>1/4</td>
<td>1/8</td>
<td>1/2</td>
</tr>
<tr>
<td>5</td>
<td>1/4</td>
<td>1/8</td>
<td>1/8</td>
<td>1/2</td>
</tr>
</tbody>
</table>
g($y_j$) 3/8 3/8 1/4 1

Find the
a) expectation of $X$; $E(X)$.
b) expectation of $Y$; $E(Y)$.
c) joint expectation of $X$ and $Y$; $E(XY)$.
d) covariance of $X$ and $Y$; $\text{cov}(X,Y)$.
e) correlation of $X$ and $Y$; $\rho(X,Y)$, and identify it.

6. Given the following discrete joint distribution of $X$ and $Y$.

<table>
<thead>
<tr>
<th>$Y(S)$</th>
<th>$X(S)$</th>
<th>-2</th>
<th>-1</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.2</td>
<td>0</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>0.1</td>
<td>$k$</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

a) Evaluate $k$. b) Find $P(Y < 4)$.
c) Find $P(X + Y < 3)$.
d) Find the expectation of $X$; $E(X)$.
e) Find the expectation of $Y$; $E(Y)$.
f) Find the joint expectation of $X$ and $Y$; $E(XY)$.
g) Find the variance of $X$; $\sigma^2_X$.
h) Find the variance of $Y$; $\sigma^2_Y$.
i) Find the covariance of $X$ and $Y$; $\text{cov}(X,Y)$.
j) Find the correlation of $X$ and $Y$; $\rho(X,Y)$, and identify it.
k) Determine whether $X$ and $Y$ are independent or not.

7. Given the following discrete joint probability distribution

\[ h(x,y) = \frac{x + y}{30}, \text{for } x = 0,1,2,3 \text{ and } y = 0,1,2. \]

Find
a) $P(X \leq 2, Y = 1)$. b) $P(X > 2, Y \leq 1)$. c) $P(X + Y = 4)$ d) $P(Y < X)$.

8. Suppose $X$ and $Y$ are two independent discrete random variables with the following distributions:

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$f(x_i)$</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.7</td>
<td>0.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$y_j$</th>
<th>$g(y_j)$</th>
<th>-2</th>
<th>5</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.3</td>
<td>0.5</td>
<td>0.2</td>
</tr>
</tbody>
</table>

a) Construct the joint distribution of $X$ and $Y$; $h(x_i,y_j)$, with marginal distributions.
b) Show that $\text{cov}(X,Y) = 0$.

9. 2 cards are selected at random from a box which contains five cards numbered 1, 1, 2, 2 and 3. Let $X(a,b) = \max(a,b)$ and $Y(a,b) = a + b$.

a) Determine the joint distribution of $X$ and $Y$, with marginal distributions.
b) Find the joint expectation of $X$ and $Y$; $E(XY)$.
c) Find the covariance of $X$ and $Y$; $\text{cov}(X,Y)$.
d) Find the correlation coefficient of $X$ and $Y$; $\rho(X,Y)$, and identify it.
10. Let $X$ and $Y$ be two discrete random variables having the following joint probability distribution

<table>
<thead>
<tr>
<th>$Y(\mathbf{s})$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$f(x_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X(\mathbf{s})$</td>
<td>1</td>
<td>0.05</td>
<td>0.05</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.05</td>
<td>0.10</td>
<td>0.35</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0</td>
<td>0.20</td>
<td>0.10</td>
</tr>
<tr>
<td>$g(y_j)$</td>
<td>0.10</td>
<td>0.35</td>
<td>0.55</td>
<td>1</td>
</tr>
</tbody>
</table>

a) Determine whether $X$ and $Y$ are independent or not.
b) Find $P(X + Y = 4)$.
c) Find $P(X \leq 2; Y \geq 2)$.
d) Find $P(X < Y)$.

e) Find the marginal density of $X$.
f) Find the marginal density of $Y$.
g) Find the expectation of $X$.
h) Find the expectation of $Y$.
i) Find the joint expectation of $X$ and $Y$.
j) Find the covariance of $X$ and $Y$; $\text{cov}(X, Y)$.
k) Find the correlation of $X$ and $Y$; $\rho(X, Y)$, and identify it.

11. A box contains 3 oranges, 2 apples and 3 bananas. A sample of 4 pieces of fruit is selected at random. If $X$ is the the number of oranges and $Y$ is the number of apples;
a) find the joint probability distribution of $X$ and $Y$.
b) find $P(X + Y \leq 2)$.

c) Find $P(X \geq 2)$.

d) Let $X$ be the number of kings, and $Y$ the number of jacks.
a) Construct the joint probability distribution of $X$ and $Y$.
b) Find the marginal distributions of $X$ and $Y$.
c) Find $P(X + Y \geq 2)$.

12. 3 cards are selected at random from 12 faced cards ((jacks, queens and kings) of an ordinary deck of 52 playing cards. Let $X$ be the number of kings, and $Y$ number of jacks.
a) Construct the joint probability distribution of $X$ and $Y$.
b) Find the marginal distributions of $X$ and $Y$.
c) Find $P(X + Y \geq 2)$.

13. Let $X$ and $Y$ be two continuous random variables with the following joint density;

$$h(x, y) = \begin{cases} 
  kxy, & 0 < x < 1 \text{ and } 0 < y < 1 \\
  0, & \text{ elsewhere}
\end{cases}$$

a) Evaluate $k$.
b) Find $P(X < 2/3; Y > 3/4)$.
c) Find $P(1/4 < X < 1/2; 1/2 < Y < 3/4)$.

14. Let $X$ and $Y$ be two continuous random variables with the following joint density;

$$h(x, y) = \begin{cases} 
  k(x+2y), & 0 < x < 1 \text{ and } 0 < y < 1 \\
  0, & \text{ elsewhere}
\end{cases}$$

a) Evaluate $k$.
b) Find $P(X < 1/2; Y > 1/2)$.
c) Find the marginal density of $X$.
d) Find the marginal density of $Y$.
e) Find the expectation of $X$; $E(X)$.
f) Find the expectation of $Y$; $E(Y)$.
g) Find the joint expectation of $X$ and $Y$; $E(XY)$.
h) Find the covariance of $X$ and $Y$; $\text{cov}(X, Y)$.
i) Find the correlation of $X$ and $Y$; $\rho(X, Y)$, and identify it.

15. Let $X$ and $Y$ be two continuous random variables with the following joint density;

$$h(x, y) = \begin{cases} 
  1, & 0 < x < y < 1 \\
  y, & \text{ elsewhere}
\end{cases}$$

Find
a) $P(X > 1/2; Y < 3/4)$.
b) $P(X + Y > 1/2)$.
c) the marginal distribution of $X$.
d) the marginal distribution of $Y$.
e) the expectation of $X$; $E(X)$.
f) the expectation of $Y$; $E(Y)$.
g) the joint expectation of $X$ and $Y$; $E(XY)$.
h) the covariance of $X$ and $Y$; $\text{cov}(X, Y)$.

16. Let $X$ and $Y$ be two continuous random variables with the following joint density;

$$h(x, y) = \begin{cases} 
  \frac{24}{7}xy, & 0 < x < 1; 0 < y < 1 \text{ and } x + y < 1 \\
  0, & \text{ elsewhere}
\end{cases}$$

Find
a) $E(X+Y)$
b) $E(X - Y)$
c) $E(XY)$

17. Let $X$ and $Y$ be two continuous random variables with the following joint density;

$$h(x, y) = \begin{cases} 
  \frac{2}{7}(x+y), & 0 < x < 1; 1 < y < 2 \\
  0, & \text{ elsewhere}
\end{cases}$$

Find
a) $E(X+Y)$
b) $E(X - Y)$
c) $E(XY)$

18. Let $X$ and $Y$ be two continuous random variables with the following joint density;

$$h(x, y) = \begin{cases} 
  2, & 0 < x < y < 1 \\
  0, & \text{ elsewhere}
\end{cases}$$

Find
a) $E(X+Y)$
b) $E(X - Y)$
c) $E(XY)$

19. Let $X$ and $Y$ be two continuous random variables with the following joint density;

$$h(x, y) = \begin{cases} 
  e^{-(x+y)}, & x > 0, y > 0 \\
  0, & \text{ elsewhere}
\end{cases}$$

Find
a) $E(X+Y)$
b) $E(X - Y)$
c) $E(XY)$

Math322 – Probability and Statistical Methods; Problem Set: 4