1. Let $x, y$ be two real numbers. Compute the determinant

\[
\begin{vmatrix}
  x & y & -x & -y \\
  y & -x & -y & x \\
  -x & -y & x & y \\
  -y & x & y & -x
\end{vmatrix}
\]

**Solution:**

\[
\begin{vmatrix}
  x & y & -x & -y \\
  y & -x & -y & x \\
  -x & -y & x & y \\
  -y & x & y & -x
\end{vmatrix}
\]

\[R_3 + R_1 \rightarrow R_1\]

\[
\begin{vmatrix}
  1 & 0 & 1 \\
  0 & 0 & 0 \\
  y & -x & y \\
  -x & y & x \\
  -y & x & -y
\end{vmatrix} = 0.
\]

2. Compute the determinant

\[
\begin{vmatrix}
  1 & 0 & 1 \\
  1 & 2 & k \\
  k & k & 6
\end{vmatrix}
\]

**Solution:**

\[
\begin{vmatrix}
  1 & 0 & 1 \\
  0 & 2 & k - 1 \\
  k & k & 6 - k
\end{vmatrix} = 2\left(6 - \frac{k}{2} - \frac{k^2}{2}\right) = 12 - k^2.
\]

3. Show that

\[
\begin{vmatrix}
  1 & 1 & 1 \\
  a & b & c \\
  a^3 & b^3 & c^3
\end{vmatrix} = (a + b + c)(c - a)(c - b)(b - a), \text{ for all } a, b, c \in R.
\]

**Solution:** Indeed, we have successively:

\[
\begin{vmatrix}
  1 & 1 & 1 \\
  a & b & c \\
  a^3 & b^3 & c^3
\end{vmatrix} = \begin{vmatrix}
  1 & 1 & 1 \\
  a - c & b - c & 0 \\
  a(a^2 - c^2) & b(b^2 - c^2) & 0
\end{vmatrix} = \begin{vmatrix}
  a - c & b - c \\
  a(a - c)(a + c) & b(b - c)(b + c)
\end{vmatrix}
\]

\[
= (a - c)(b - c) \begin{vmatrix}
  1 & 1 \\
  a(a + c) & b(b + c)
\end{vmatrix}
\]

\[
= (a - c)(b - c)(b^2 + bc - a^2 - ac)
\]

\[
= (a - c)(b - c)(b - a)(b + a) + (b - a)c
\]

\[
= (a - b)(b - c)(c - a)(a + b + c) = (a + b + c)V(a, b, c).
\]
4. Show that: \[
\begin{vmatrix}
\alpha^2 & \beta^2 & \gamma^2 \\
\alpha^3 & \beta^3 & \gamma^3 \\
\end{vmatrix} = (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(\alpha\beta + \beta\gamma + \gamma\alpha) \text{ for all } \alpha, \beta, \gamma \in \mathbb{R};
\]

**Solution:** Indeed, we have successively:

\[
\begin{aligned}
&\begin{vmatrix}
\alpha^2 & \beta^2 & \gamma^2 \\
\alpha^3 & \beta^3 & \gamma^3 \\
\end{vmatrix} = \begin{vmatrix}
\alpha^2 - \gamma^2 & \beta^2 - \gamma^2 & 0 \\
\alpha^2(\alpha - \gamma) & \beta^2(\beta - \gamma) & 0 \\
\end{vmatrix} \\
&= (\alpha - \gamma)(\alpha + \gamma)(\beta - \gamma)(\beta + \gamma) \\
&= (\alpha - \gamma)(\beta - \gamma)(\alpha + \gamma) \beta \gamma + (\beta - \alpha)(\beta + \alpha) \\
&= (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(\alpha\beta + \beta\gamma + \gamma\alpha) = (\alpha\beta + \beta\gamma + \gamma\alpha)V(\alpha, \beta, \gamma)
\end{aligned}
\]

5. Find the inverse of the matrix \( A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}. \)

**Solution:** By performing row-reduction operations on the matrix \( \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{pmatrix} \) we have successively:

\[
\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{pmatrix} \rightarrow
\]
\[
\begin{bmatrix}
1 & 2 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-14 & 6 & 3 \\
13 & -5 & -3 \\
5 & -2 & -1
\end{bmatrix}
\]
\[
R_1 \rightarrow -2R_2 + R_1 \\
0 & 1 & 0 \\
0 & 0 & 1
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-40 & 16 & 9 \\
13 & -5 & -3 \\
5 & -2 & -1
\end{bmatrix}
\]

6. Find the inverse of the matrix \( A = \begin{bmatrix}
-2 & -1 & 4 \\
3 & 1 & -7 \\
2 & 0 & -5
\end{bmatrix}\) and write \( A^{-1} \) as a product of elementary matrices.

Solution:
\[
\begin{bmatrix}
-2 & -1 & 4 \\
3 & 1 & -7 \\
2 & 0 & -5
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[
r_2 + r_1 \rightarrow r_1 \\
3 & 1 & -7 \\
2 & 0 & -5
\]
\[
\begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[
-3r_1 + r_2 \rightarrow r_2 \\
3 & 1 & -7 \\
2 & 0 & -5
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & 2 & -2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[
3r_3 + r_1 \rightarrow r_3 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\]
\[
\]

Therefore \( E_5, E_4, E_3, E_2, E_1, A = I_3 \), or equivalently \( A^{-1} = E_5, E_4, E_3, E_2, E_1 \), where

\[
I_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
7. Express the matrix
\[ A = \begin{bmatrix}
0 & 1 & 7 & 8 \\
1 & 3 & 3 & 8 \\
-2 & -5 & 1 & -8 \\
\end{bmatrix} \]
in the form \( A = EFGR \), where \( E, F, G \) are elementary matrices and \( R \) in row-echelon form.

**Solution:**

\[
A = \begin{bmatrix}
0 & 1 & 7 & 8 \\
1 & 3 & 3 & 8 \\
-2 & -5 & 1 & -8 \\
\end{bmatrix}
\]

\[
\begin{array}{c}
 r_1 \leftrightarrow r_2 \\
2r_1 + r_3 \rightarrow r_3
\end{array}
\]

\[
\begin{array}{c}
1 & 3 & 3 & 8 \\
0 & 1 & 7 & 8 \\
0 & 1 & 7 & 8
\end{array}
\] = \( R \)-matrix in row-echelon form.

Consequently \( R = E_3 E_2 E_1 A \), or equivalently \( A = E_1^{-1} E_2^{-1} E_3^{-1} R \), where

\[
I_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{array}{c}
r_1 \leftrightarrow r_2 \\
2r_1 + r_3 \rightarrow r_3
\end{array}
\]

\[
\begin{array}{c}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}
\] = \( E_1 = E_1^{-1} \)

\[
I_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{array}{c}
2r_1 + r_3 \rightarrow r_3
\end{array}
\]

\[
\begin{array}{c}
1 & 0 & 0 \\
0 & 1 & 0 \\
2 & 0 & 1
\end{array}
\] \( = E_2 \)

\[
I_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{array}{c}
-r_2 + r_3 \rightarrow r_3
\end{array}
\]

\[
\begin{array}{c}
1 & 0 & 0 \\
0 & 1 & 0 \\
-2 & 0 & 1
\end{array}
\] \( = E_2^{-1} \)

\[
I_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{array}{c}
-r_2 + r_3 \rightarrow r_3
\end{array}
\]

\[
\begin{array}{c}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}
\] \( = E_3 \)

Now it is enough to take \( E = E_1^{-1}, F = E_2^{-1}, G = E_3^{-1} \).

8. Show that

\[
\begin{vmatrix}
\left| \begin{array}{cccc}
a_1 + c_1 & a_2 + c_2 & a_3 + c_3 & a_4 + c_4 \\
b_1 + d_1 & b_2 + d_2 & b_3 + d_3 & b_4 + d_4 \\
a_1 + d_1 & a_2 + d_2 & a_3 + d_3 & a_4 + d_4 \\
b_1 + c_1 & b_2 + c_2 & b_3 + c_3 & b_4 + c_4
\end{array} \right| = 0,
\end{vmatrix}
\]
for all $a_1, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4 \in \mathbb{R}$.

**Solution:** Indeed, we have successively:

\[
\begin{vmatrix}
  a_1 + c_1 & a_2 + c_2 & a_3 + c_3 & a_4 + c_4 \\
  b_1 + d_1 & b_2 + d_2 & b_3 + d_3 & b_4 + d_4 \\
  a_1 + d_1 & a_2 + d_2 & a_3 + d_3 & a_4 + d_4 \\
  b_1 + c_1 & b_2 + c_2 & b_3 + c_3 & b_4 + c_4
\end{vmatrix}
= \begin{vmatrix}
  1 & 0 & 1 & 0 \\
  0 & 1 & 0 & 1 \\
  1 & 0 & 0 & 1 \\
  0 & 1 & 1 & 0
\end{vmatrix}
\cdot
\begin{vmatrix}
  a_1 & a_2 & a_3 & a_4 \\
  b_1 & b_2 & b_3 & b_4 \\
  c_1 & c_2 & c_3 & c_4 \\
  d_1 & d_2 & d_3 & d_4
\end{vmatrix} = 0
\]

9. Consider the matrix $A = \begin{bmatrix}
  1 & 1 & 1 & 1 \\
  x & y & z & w \\
  w & x & y & z \\
  y + z & z + w & w + x & x + y
\end{bmatrix}$. Show that $\det(A) = 0$ for all $x, y, z, w \in \mathbb{R}$.

**Solution:** Indeed, for all $x, y, z, w \in \mathbb{R}$ we have successively:

\[
\det(A) = \begin{vmatrix}
  1 & 1 & 1 & 1 \\
  x & y & z & w \\
  w & x & y & z \\
  y + z & z + w & w + x & x + y
\end{vmatrix}
\]

\[
\overset{r_3 + r_4 \rightarrow r_4}{\Rightarrow}
\begin{vmatrix}
  1 & 1 & 1 & 1 \\
  x & y & z & w \\
  w & x & y & z \\
  x + y + z + w & x + y + z + w & y + z + w + x & z + w + x + y
\end{vmatrix}
\]

\[
\overset{r_2 + r_4 \rightarrow r_4}{\Rightarrow}
\begin{vmatrix}
  1 & 1 & 1 & 1 \\
  x & y & z & w \\
  w & x & y & z \\
  x + y + z + w & x + y + z + w & y + z + w + x & z + w + x + y
\end{vmatrix} = (x + y + z + w)
\]

\[
\begin{vmatrix}
  1 & 1 & 1 & 1 \\
  x & y & z & w \\
  w & x & y & z \\
  1 & 1 & 1 & 1
\end{vmatrix} = 0.
\]

10. Show that $V(a_1, a_2, \ldots, a_n) = \prod_{1 \leq i < j \leq n} (a_j - a_i) = (-1)^{\frac{n(n-1)}{2}} \prod_{1 \leq i < j \leq n} (a_i - a_j)$, where $V(a_1, a_2, \ldots, a_n)$ is the Vandermonde determinant

\[
\begin{vmatrix}
  1 & 1 & \cdots & 1 \\
  a_1 & a_2 & \cdots & a_n \\
  a_1^2 & a_2^2 & \cdots & a_n^2 \\
  \vdots & \vdots & \ddots & \vdots \\
  a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1}
\end{vmatrix}
\]
Solution: Indeed, we have successively:

\[
V(a_1, a_2, \ldots, a_n) = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
a_1 & a_2 & \cdots & a_n \\
a_1^2 & a_2^2 & \cdots & a_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \\
\end{vmatrix}
\]

\[
= (-1)^{n-1}(a_1 - a_n)(a_2 - a_n) \cdots (a_{n-1} - a_n)
\]

\[
= \left[ (-1)^{n-1} \prod_{i=1}^{n-1} (a_i - a_n) \right] V(a_1, a_2, \ldots, a_{n-1})
\]

\[
= \left[ \prod_{i=1}^{n-1} (a_n - a_i) \right] V(a_1, a_2, \ldots, a_{n-1})
\]

\[
= \left[ \prod_{i=1}^{n-1} (a_n - a_i) \right] \left[ \prod_{i=1}^{n-2} (a_{n-1} - a_i) \right] V(a_1, a_2, \ldots, a_{n-2})
\]

\[
\vdots
\]

\[
= \prod_{1 \leq i < j \leq n} (a_j - a_i)
\]

\[
= (-1)^{\frac{n(n-1)}{2}} \prod_{1 \leq i < j \leq n} (a_i - a_j)
\]

11. For which values of \(x \in \mathbb{R}\), is the matrix

\[
A = \begin{bmatrix}
2 & \frac{x}{2} & 0 \\
\frac{x}{2} & \frac{1}{4} & \frac{x}{2} \\
0 & \frac{x}{2} & 2
\end{bmatrix}
\]

invertible?
Solution: \[ \det(A) = \begin{vmatrix} 2 & \frac{x}{2} & 0 \\ \frac{x}{2} & \frac{1}{x} & \frac{x}{2} \\ 0 & \frac{x}{2} & 2 \end{vmatrix} = 1 - \frac{x^2}{2} - \frac{x^2}{2} = 1 - x^2. \] We now recall that \( A \) is invertible if and only if \( \det(A) \neq 0 \iff 1 - x^2 \neq 0 \iff x^2 \neq 1 \iff x \neq \pm 1. \)