ITERATION FOR NONLINEAR SYSTEMS

Many problems in engineering and science require the solution of a system of nonlinear equations. Consider a system of two nonlinear equations.

\[ f(x, y) = 0 \]
\[ g(x, y) = 0 \]

The problem can be stated as follows

Given the continuous functions \( f(x, y) \) and \( g(x, y) \), find the values \( x = x^* \) and \( y = y^* \) such that \( f(x^*, y^*) = 0 \) and \( g(x^*, y^*) = 0 \).

FIXED POINT ITERATION

Given

\[ f_1(x, y) = 0 \]
\[ f_2(x, y) = 0 \]

for applying the method of iteration the system (1) is reduced to the form

\[ x = g_1(x, y) \]
\[ y = g_2(x, y) \]

The algorithm of solution is given by the formulas

\[ x_{n+1} = g_1(x_n, y_n) \]
\[ y_{n+1} = g_2(x_n, y_n) \]

Where \( x_0, y_0 \) is some initial approximation.

Theorem: Let is some closed neighbourhood \( R (a \leq x \leq A, b \leq y \leq B) \) there be one and only one solution \( x = \xi, \ y = \eta \) for the system (2) if,

1. The functions \( g_1(x, y) \) and \( g_2(x, y) \) are defined and continuously differentiable in \( R \).
2. The initial approximations \( x_0, y_0 \) and all successive approximations \( x_n, y_n \) (\( n = 1, 2, \ldots \)) belong to \( R \).
3. The following inequalities are fulfilled in $R$

\[
\left| \frac{\partial g_1}{\partial x} \right| + \left| \frac{\partial g_1}{\partial y} \right| < 1 \]
\[
\left| \frac{\partial g_2}{\partial x} \right| + \left| \frac{\partial g_2}{\partial y} \right| < 1 \] 

\text{.............(4)}

or $\|J(G(x,y))\| < 1$

Then the process of successive approximations (3) converges to the solution $x = \xi$, $y = \eta$ of the system.

Constructing iterative functions for the system

Consider,

\[
g_1(x,y) = x + \alpha f_1(x,y) + \beta f_2(x,y) \]
\[
g_2(x,y) = y + \gamma f_1(x,y) + \delta f_2(x,y) \]

where, $\alpha \delta \neq \beta \gamma$

find the coefficients $\alpha, \beta, \gamma$ \textit{and} $\delta$ as approximate solutions of the following system of equations at $(x_0, y_0)$.

\[
\frac{\partial g_1}{\partial x} = 0 \quad \Rightarrow \quad 1 + \alpha \frac{\partial f_1}{\partial x} + \beta \frac{\partial f_2}{\partial x} = 0
\]
\[
\frac{\partial g_1}{\partial y} = 0 \quad \Rightarrow \quad \alpha \frac{\partial f_1}{\partial y} + \beta \frac{\partial f_2}{\partial y} = 0
\]
\[
\frac{\partial g_2}{\partial x} = 0 \quad \Rightarrow \quad \gamma \frac{\partial f_1}{\partial x} + \delta \frac{\partial f_2}{\partial x} = 0
\]
\[
\frac{\partial g_2}{\partial y} = 0 \quad \Rightarrow \quad 1 + \gamma \frac{\partial f_1}{\partial y} + \delta \frac{\partial f_2}{\partial y} = 0
\]
Example: Perform 2 iterations of a convergent fixed point method to approximate the root of the following system.

\[ f_1(x, y) = \sqrt{x + y} + xy - 2.3 = 0 \]  \hspace{1cm} (1)  
\[ f_2(x, y) = x^3 - y^3 - 9xy = 0 \]

by taking \((x_0, y_0) = (1.92, 0.44)\).

Solution:

Construct iterative function \(g_1(x, y)\) and \(g_2(x, y)\) where,

\[ g_1(x, y) = x + \alpha(\sqrt{x + y} + xy - 2.3) + \beta(x^3 - y^3 - 9xy) \]  \hspace{1cm} (2)  
\[ g_2(x, y) = y + \gamma(\sqrt{x + y} + xy - 2.3) + \delta(x^3 - y^3 - 9xy) \]

\[ \frac{\partial g_1}{\partial x} = 0 \Rightarrow 1 + \alpha \left( \frac{1}{2\sqrt{x + y}} + y \right) + \beta \left( 3x^2 - 9y \right) = 0 \]  \hspace{1cm} (3)  
\[ \frac{\partial g_1}{\partial y} = 0 \Rightarrow \alpha \left( \frac{1}{2\sqrt{x + y}} + x \right) + \beta \left( -3y^2 - 9x \right) = 0 \]  \hspace{1cm} (4)  
\[ \frac{\partial g_2}{\partial x} = 0 \Rightarrow \gamma \left( \frac{1}{2\sqrt{x + y}} + y \right) + \delta \left( 3x^2 - 9y \right) = 0 \]  \hspace{1cm} (5)  
\[ \frac{\partial g_2}{\partial y} = 0 \Rightarrow 1 + \gamma \left( \frac{1}{2\sqrt{x + y}} + x \right) + \delta \left( -3y^2 - 9x \right) = 0 \]  \hspace{1cm} (6)

where,  
\((x_0, y_0) = (1.92, 0.44)\) substitute in (3), (4), (5) and (6)

from (3) and (4)

\[ 17.86 \times (0.765\alpha + 7.0992\beta = -1) \]  
\[ 7.0992 \times (2.245\alpha - 17.86\beta = 0) \]

\[ 29.6\alpha = -17.86 \Rightarrow \alpha = -0.603 \]  
\[ \beta = 0.0757 \]
from (5) and (6)

\[
\begin{align*}
17.86 \times (0.765\gamma + 7.0992\delta) &= 0 \\
7.0992 \times (2.245\gamma - 17.86\delta) &= -1
\end{align*}
\]
\[
\text{\underline{\text{\tiny +}}} \quad 29.6\gamma = -7.0992 \quad \Rightarrow \gamma = -0.239 \\
\delta = 0.0258
\]

then,

\[
g_1(x, y) = x - 0.603(\sqrt{x + y + xy - 2.3} - 0.0757(x^3 - y^3 - 9xy) \\
g_2(x, y) = y - 0.239(\sqrt{x + y + xy - 2.3} + 0.0258(x^3 - y^3 - 9xy)
\]

such that \( \|J(G(x, y))\| < 1 \)

where

\[
x_{n+1} = g_1(x_n, y_n) \\
y_{n+1} = g_2(x_n, y_n)
\]

Iteration 1: \((x_0, y_0) = (1.92, 0.44)\)

\[
x_1 = 1.92 - 0.603\left(\sqrt{1.92 + 0.44 + (1.92 \times 0.44) - 2.3}\right) - 0.0757\left(1.92^3 - 0.44^3 - 9(1.92 \times 0.44)\right) = 1.917 \\
y_1 = 0.44 - 239\left(\sqrt{1.92 + 0.44 + (1.92 \times 0.44) - 2.3}\right) + 0.0258\left(1.92^3 - 0.44^3 - 9(1.92 \times 0.44)\right) = 0.404
\]

Iteration 1: \((x_1, y_1) = (1.917, 0.4047)\)

\[
x_2 = 1.917 - 0.603\left(\sqrt{1.917 + 0.404 + (1.917 \times 0.404) - 2.3}\right) - 0.0757\left(1.917^3 - 0.404^3 - 9(1.917 \times 0.404)\right) = 1.91757 \\
y_2 = 0.404 - 239\left(\sqrt{1.910 + 0.404 + (1.917 \times 0.404) - 2.3}\right) + 0.0258\left(1.917^3 - 0.404^3 - 9(1.917 \times 0.404)\right) = 0.4047
\]

Check error

\[
\varepsilon = \|f_1(x_2, y_2)\|_{\infty} = \max\{|f_1(x_2, y_2)|, |f_2(x_2, y_2)|\} \\
|f_1(1.91757, 0.4047)| = 5.9818 \times 10^{-5} \\
|f_2(1.91757, 0.4047)| = 4 \times 10^{-4}
\]

\[
\varepsilon = \max\{5.9818 \times 10^{-5}, 4 \times 10^{-4}\} = 4 \times 10^{-4}
\]
Example (fixed point system) Construct the iterative functions \( g_1(x, y) \) and \( g_2(x, y) \) to approximate the root of

\[
f_1(x, y) = \cos(x + 5y) - (x + y^2) + 2.46 = 0
\]

\[
f_2(x, y) = y^2(2 - x) - x^3 = 0
\]

using fixed point iteration. Perform two iterations with \((x_0, y_0) = (0.9, 1.5)\).

Solution: where

\[
g_1(x, y) = x + \alpha \left[ \cos(x + 5y) - (x + y^2) + 2.46 \right] + \beta \left[ y^2 (2 - x) - x^3 \right] \quad \text{........(2)}
\]

\[
g_2(x, y) = y + \gamma \left[ \cos(x + 5y) - (x + y^2) + 2.46 \right] + \delta \left[ y^2 (2 - x) - x^3 \right] \quad \text{........(3)}
\]

\[
\frac{\partial g_1}{\partial x} = 0 \Rightarrow 1 + \alpha \left[ -\sin(x + 5y) - 1 \right] + \beta \left[ -y^2 - 3x^2 \right] = 0 \quad \text{........(4)}
\]

\[
\frac{\partial g_1}{\partial y} = 0 \Rightarrow \alpha \left[ -5\sin(x + 5y) - 2y \right] + \beta \left[ 4y - 2xy \right] = 0 \quad \text{........(5)}
\]

\[
\frac{\partial g_2}{\partial x} = 0 \Rightarrow \gamma \left[ -\sin(x + 5y) - 1 \right] + \delta \left[ -y^2 - 3x^2 \right] = 0 \quad \text{........(6)}
\]

\[
\frac{\partial g_2}{\partial y} = 0 \Rightarrow 1 + \gamma \left[ -5\sin(x + 5y) - 2y \right] + \delta \left[ 4y - 2xy \right] = 0 \quad \text{........(7)}
\]

Substitute \((x_0, y_0) = (0.9, 1.5)\) in (4), (5), (6) and (7) to find \(\alpha, \beta, \gamma\) and \(\delta\) (exercise)

Where \(\alpha = 0.082, \beta = 0.1811, \gamma = 0.1166\) and \(\delta = -0.0461\) then

\[
x_{n+1} = x_n + 0.082 \left[ \cos(x_n + 5y_n) - (x_n + y_n^2) + 2.46 \right] + 0.1811 \left[ y_n^2 (2 - x_n) - x_n^3 \right]
\]

\[
y_{n+1} = y_n + 0.1166 \left[ \cos(x_n + 5y_n) - (x_n + y_n^2) + 2.46 \right] - 0.0461 \left[ y_n^2 (2 - x_n) - x_n^3 \right]
\]

Start \((x_0, y_0) = (0.9, 1.5)\) to find

\[
(x_1, y_1) = (1.117, 1.278)
\]

\[
(x_2, y_2) = (1.129, 1.281) \quad \text{(exercise) also calculate } \left\| f_i(1.129, 1.281) \right\|_\infty \text{ for } i = 1, 2
\]

EXERCİSE) Solve the system

\[
(x - 1)^3 - y = 0
\]

\[
(x - 1)^2 + \left( y - \frac{1}{2} \right)^2 = 4 \quad \text{, with } (x_0, y_0) = (2.3, 1.9)
\]

by using fixed point method. Perform 2 iteration.
Newton’s Method for Nonlinear Systems

Consider the system of nonlinear equations

\[ u = f_1(x, y) \]
\[ v = f_2(x, y) \]  \hspace{1cm} (1)

Which can be considered a transformation from the xy-plane into uv-plane with starting point \((x_0, y_0)\) where image is the point \((u_0, v_0)\).

If both \(f_1(x, y)\) and \(f_2(x, y)\) have continuous partial derivatives then,

\[ u - u_0 \approx \frac{\partial f_1(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f_1(x_0, y_0)}{\partial y}(y - y_0) \]
\[ v - v_0 \approx \frac{\partial f_2(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f_2(x_0, y_0)}{\partial y}(y - y_0) \]

Then the Jacobian Matrix \(J(x_0, y_0)\) is used, this relationship is easier to visualize.

\[
\begin{pmatrix}
u - u_0 \\
v - v_0
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial f_1(x_0, y_0)}{\partial x} & \frac{\partial f_1(x_0, y_0)}{\partial y} \\
\frac{\partial f_2(x_0, y_0)}{\partial x} & \frac{\partial f_2(x_0, y_0)}{\partial y}
\end{pmatrix}
\begin{pmatrix}
(x - x_0) \\
(y - y_0)
\end{pmatrix}
\]  \hspace{1cm} (2)

If the system in (1) is written as a vector \(V = F(x)\) the Jacobian \(J(x, y)\) is the two dimensional analog of the derivative (2) can be written as

\[ \Delta F \approx J(x_0, y_0)\Delta x \]  \hspace{1cm} (3)

We now use (3) to derive Newton’s Method in two dimensions.

Consider the system (1) with \(u\) and \(v\) set equal to zero.

\[ 0 = f_1(p, q) \]
\[ 0 = f_2(p, q) \]  \hspace{1cm} (4)

Suppose that \((p, q)\) is a solution of (4).

To develop Newton’s Method for solving (4), we need to consider small changes in the functions near the point \((p_0, q_0)\).

\[ u - u_0 = \Delta u \quad \Delta p = x - p_0 \]
\[ v - v_0 = \Delta v \quad \Delta q = y - q_0 \]  \hspace{1cm} (5)

Set \((x, y) = (p, q)\) in (1) and use (4) to see that \((u, v) = (0, 0)\). Hence the changes in the dependent variable are
\[ u - u_0 = f_1(p,q) - f_1(p_0,q_0) = 0 - f_1(p_0,q_0) \]

\[ v - v_0 = f_2(p,q) - f_2(p_0,q_0) = 0 - f_2(p_0,q_0) \]

Use the result of (6) in (2) to get the linear transformation

\[
\begin{pmatrix}
\frac{\partial f_1(p_0,q_0)}{\partial x} & \frac{\partial f_1(p_0,q_0)}{\partial y} \\
\frac{\partial f_2(p_0,q_0)}{\partial x} & \frac{\partial f_2(p_0,q_0)}{\partial y}
\end{pmatrix}
\begin{pmatrix}
\Delta P \\
\Delta q
\end{pmatrix}
= -\begin{pmatrix}
f_1(p_0,q_0) \\
f_2(p_0,q_0)
\end{pmatrix}
\]

\[
J(p_0,q_0)\Delta P = -\Delta F \quad \text{solve} \quad \Delta P,
\]

\[
\Delta P = -\left[J(p_0,q_0)\right]^{-1}\Delta F
\]

OUTLINE OF NEWTON’S METHOD

Suppose that \( P_k \) as been obtained

Step 1: Evaluate the function

\[ F(P_k) = \begin{pmatrix} f_1(p_k,q_k) \\ f_2(p_k,q_k) \end{pmatrix} \]

Step 2: Evaluate the Jacobian Matrix

\[
\begin{pmatrix}
\frac{\partial f_1(p_k,q_k)}{\partial x} & \frac{\partial f_1(p_k,q_k)}{\partial y} \\
\frac{\partial f_2(p_k,q_k)}{\partial x} & \frac{\partial f_2(p_k,q_k)}{\partial y}
\end{pmatrix}
\]

Step 3: Solve

\[ J(P_k)\Delta P = -\Delta F(P_k) \quad \text{for} \quad \Delta P \]

Step 4: Compute the next point

\[ P_{k+1} = P_k + \Delta P \]

Now, repeat the process
Example: Use starting point \((p_0, q_0) = (3.795, 4.594)\) for the Newton’s method to solve the nonlinear systems

\[
\begin{align*}
  x^2 - \frac{2}{3}y^2 - \frac{1}{3} &= 0 \\
  \frac{x^2}{2} - x + y - 8 &= 0
\end{align*}
\]

Compute \((p_1, q_1)\) and \((p_2, q_2)\).

Solution: where,

\[
\begin{align*}
  f_1(x, y) &= x^2 - \frac{2}{3}y^2 - \frac{1}{3} \
  \frac{\partial f_1}{\partial x} &= 2x \
  \frac{\partial f_1}{\partial y} &= -\frac{4}{3}y \\
  f_2(x, y) &= \frac{x^2}{2} - x + y - 8 \
  \frac{\partial f_2}{\partial x} &= x - 1 \
  \frac{\partial f_2}{\partial y} &= 1
\end{align*}
\]

Iteration 1:

\[
F(3.795, 4.594) = \begin{bmatrix}
  3.795^2 - \frac{2}{3} \cdot 4.594^2 - \frac{1}{3} \\
  \frac{3.795^2}{2} - 3.795 + 4.594 - 8
\end{bmatrix} = \begin{bmatrix}
  -1.19 \times 10^{-3} \\
  1.25 \times 10^{-5}
\end{bmatrix}
\]

\[
J(3.795, 4.594) = \begin{bmatrix}
  2(3.795) & -4 \cdot (4.594) \\
  3.795 - 1 & 1
\end{bmatrix} = \begin{bmatrix}
  7.59 & -6.125 \\
  2.795 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
  7.59 \\
  2.795
\end{bmatrix} \begin{bmatrix}
  \Delta p \\
  \Delta q
\end{bmatrix} = \begin{bmatrix}
  -1.19 \times 10^{-3} \\
  1.25 \times 10^{-5}
\end{bmatrix}
\]

\[
7.59 \Delta p - 6.125 \Delta q = 1.19 \times 10^{-3}
\]

\[
2.795 \Delta p + \Delta q = -1.25 \times 10^{-5} \quad \text{(exercise)}
\]

where, \(\Delta p = 4.50613 \times 10^{-5}\) , \(\Delta q = -1.38446 \times 10^{-4}\)

\[
\Delta p = p_1 - p_0 \quad \Rightarrow \quad p_1 = \Delta p + p_0 = 4.50613 \times 10^{-5} + 3.795 \approx 3.7951
\]

\[
\Delta q = q_1 - q_0 \quad \Rightarrow \quad q_1 = \Delta q + q_0 = -1.38446 \times 10^{-4} + 4.594 \approx 4.59386
\]

\((p_1, q_1) = (3.7951, 4.59386)\)
Iteration 2:

\[ F(3.7951, 4.59386) = \begin{pmatrix}
\frac{3.7951^2}{3} - \frac{2}{3} 4.59386^2 - \frac{1}{3} \\
\frac{3.7951^2}{2} - 3.7951 + 4.59386 - 8
\end{pmatrix} = \begin{pmatrix}
2.01 \times 10^{-3} \\
-1.079 \times 10^{-4}
\end{pmatrix} \]

\[ J(3.7951, 4.59386) = \begin{pmatrix}
2(3.7951) & \frac{-4}{3} (4.59386) \\
3.7951 - 1 & 1
\end{pmatrix} = \begin{pmatrix}
7.5902 & -6.1248 \\
2.7951 & 1
\end{pmatrix} \]

\[
\begin{pmatrix}
7.5902 & -6.1248 \\
2.7951 & 1
\end{pmatrix}
\begin{pmatrix}
\Delta p \\
\Delta q
\end{pmatrix}
= \begin{pmatrix}
2.01 \times 10^{-3} \\
-1.079 \times 10^{-4}
\end{pmatrix}
\]

\[ 7.5902 \Delta p - 6.1248 \Delta q = -2.01 \times 10^{-3} \]

\[ 2.7951 \Delta p + \Delta q = 1.079 \times 10^{-4} \quad \text{(exercise)} \]

\[ \Delta p = p_2 - p_1 \quad \Rightarrow \quad p_2 = \Delta p + p_1 = 3.795 \]

\[ \Delta q = q_2 - q_1 \quad \Rightarrow \quad q_2 = \Delta q + q_1 = 4.5935 \]

\[ (p_2, q_2) = (3.795, 4.5935) \]