Question 1:
Evaluate
(a) \( \int_{1}^{3} \int_{2}^{4} (40 - 2xy) \, dy \, dx \)
(b) \( \int_{2}^{4} \int_{1}^{3} (40 - 2xy) \, dx \, dy \)

Solution
\[
\int_{1}^{3} \int_{2}^{4} (40 - 2xy) \, dy \, dx = \int_{1}^{3} \left[ \int_{2}^{4} (40 - 2xy) \, dy \right] \, dx \\
= \int_{1}^{3} \left[ (40y - xy^2) \right]_{y=2}^{4} \, dx \\
= \int_{1}^{3} [(160 - 16x) - (80 - 4x)] \, dx \\
= \int_{1}^{3} (80 - 12x) \, dx \\
= (80x - 6x^2)|_1^3 = 112 \\
\]

\[
\int_{2}^{4} \int_{1}^{3} (40 - 2xy) \, dx \, dy = \int_{2}^{4} \left[ \int_{1}^{3} (40 - 2xy) \, dx \right] \, dy \\
= \int_{2}^{4} \left[ (40x - x^2y) \right]_{y=1}^{3} \, dy \\
= \int_{2}^{4} [(120 - 9y) - (40 - y)] \, dy \\
= \int_{2}^{4} (80 - 8y) \, dy \\
= (80y - 4y^2)|_2^4 = 112
\]

Question 3:
Choosing a convenient order of integration, evaluate \( \iint_{R} ye^y \, dA \), where \( R = \{(x,y); 0 \leq x \leq 1, 0 \leq y \leq \ln 2 \} \).

Solution
The iterated integral \( \int_{0}^{\ln 2} \int_{0}^{1} ye^y \, dx \, dy \) requires first integrating \( ye^y \) with respect to \( y \), which entails integration by parts. An easier approach is to integrate first with respect to \( x \):
\[
\int_{0}^{\ln 2} \int_{0}^{1} ye^y \, dx \, dy = \int_{0}^{\ln 2} \left[ (e^x) \right]_0^1 \, dy
\]
Evaluate the inner integral with respect to \( x \):
\[
\int_{0}^{\ln 2} (e^1 - 1) \, dy
\]
Simplify:
\[
= (e - 1)|_0^{\ln 2}
\]
Evaluate the outer integral with respect to \( y \):
\[
= 1 - \ln 2
\]
Simplify:

Question 4:
Use an iterated integral to find the area of the region bounded by the graphs of the equations.
\( \sqrt{x} + \sqrt{y} = 2 \), \( x = 0 \), \( y = 0 \)

1. \( \int_{0}^{4} \int_{0}^{2 - \sqrt{x}} y \, dy \, dx = \int_{0}^{4} \left[ \frac{(2 - \sqrt{x})^2}{2} \right]_0^x \, dx \\
= \left[ \frac{8}{3} \right]_0^4 \\
= \frac{8}{3}
\]

2. \( \int_{0}^{4} (4 - 4 \sqrt{x} + x) \, dx \\
= \left[ 4x - \frac{8}{3} \sqrt{x} + \frac{x^2}{2} \right]_0^4 \\
= \frac{8}{3}
\]

3. \( \int_{0}^{4} \int_{0}^{2 - \sqrt{x}} dy \, dx = \frac{8}{3} \\
\]

6. Integration steps are similar to those above.

7. [Graph of \( y = (2 - \sqrt{x})^2 \)]
Question 5:

Use an iterated integral to find the area of the region.

\[
y = 4 - x^2 \\
y = x + 2
\]

1. \( y = 4 - x^2 \)
2. \( y = x + 2 \)

2. \( A = \int_{-2}^{1} \int_{-2}^{4-x^2} \; dy \; dx \)
3. \( = \int_{-2}^{1} \left[ y \right]_{-2}^{4-x^2} \; dx \)
4. \( = \int_{-2}^{1} (4 - x^2 - x - 2) \; dx \)
5. \( = \int_{-2}^{1} (2 - x - x^2) \; dx \)
6. \( = \left[ 2x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_{-2}^{1} \)
7. \( = \frac{9}{2} \)
8. \( A = \int_{0}^{3} \int_{0}^{\sqrt[3]{y}} dx \; dy + 2 \int_{3}^{4} \int_{0}^{\sqrt[4]{4-y}} dx \; dy \)
9. \( = \int_{0}^{3} \left[ x \right]_{0}^{\sqrt[3]{y}} \; dy + 2 \int_{3}^{4} \left[ x \right]_{0}^{\sqrt[4]{4-y}} \; dy \)
10. \( = \int_{0}^{3} (y - 2 + \sqrt[4]{4-y}) \; dy + 2 \int_{3}^{4} \sqrt[4]{4-y} \; dy \)
11. \( = \left[ \frac{1}{2}y^2 - 2y - 2 \frac{(4-y)^{3/2}}{3} \right]_{0}^{3} - \left[ \frac{4}{3}(4-y)^{3/2} \right]_{3}^{4} \)
12. \( = \frac{9}{2} \)

Question 6:

Sketch the region \( R \) of integration and switch the order of integration.

\[
\int_{1}^{10} \int_{0}^{\ln y} f(x, y) \; dx \; dy
\]

1. [Diagram]
2. \( \int_{1}^{10} \int_{0}^{\ln y} f(x, y) \; dx \; dy, \; 0 \leq x \leq \ln y, \; 1 \leq y \leq 10 \)
3. \( = \int_{0}^{10} \int_{e^x}^{10} f(x, y) \; dy \; dx \)

Question 7:

Sketch the region \( R \) of integration and switch the order of integration.

\[
\int_{-1}^{1} \int_{x^2}^{1} f(x, y) \; dy \; dx
\]

1. [Diagram]
2. \( \int_{-1}^{1} \int_{x^2}^{1} f(x, y) \; dy \; dx, \; x^2 \leq y \leq 1, \; -1 \leq x \leq 1 \)
3. \( = \int_{0}^{1} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y) \; dx \; dy \)
Question: 8

Use an iterated integral to find the area of the region bounded by the graphs of the equations.

\[2x - 3y = 0, \quad x + y = 5, \quad y = 0\]

1. \[A = \int_{0}^{3} \int_{0}^{2x/3} dy \, dx + \int_{3}^{5} \int_{0}^{5-x} dy \, dx\]

2. \[= \int_{0}^{3} \left[ y \right]_{0}^{2x/3} dx + \int_{3}^{5} \left[ y \right]_{0}^{5-x} dx\]

3. \[= \int_{0}^{3} \frac{2x}{3} \, dx + \int_{3}^{5} (5 - x) \, dx\]

4. \[= \left[ \frac{1}{3}x^2 \right]_{0}^{3} + \left[ 5x - \frac{1}{2}x^2 \right]_{3}^{5}\]

5. \[= 5\]

6. \[A = \int_{0}^{3} \int_{y/2}^{5-y} dx \, dy\]

7. \[= \int_{0}^{3} \left[ x \right]_{y/2}^{5-y} dy\]

8. \[= \int_{0}^{3} \left( 5 - y - \frac{3y}{2} \right) \, dy\]

9. \[= \int_{0}^{3} \left( 5 - \frac{5y}{2} \right) \, dy\]
Question 9

Area of a plane region  Find the area of the region $R$ bounded by $y = x^2$, $y = -x + 12$, and $y = 4x + 12$ (Figure 13.26).

Solution

The region $R$ in its entirety is bounded neither above and below by two curves, nor on the left and right by two curves. However, when decomposed along the $y$-axis, $R$ may be viewed as two regions $R_1$ and $R_2$ that are each bounded above and below by a pair of curves. Notice that the parabola $y = x^2$ and the line $y = -x + 12$ intersect in the first quadrant at the point $(3, 9)$, while the parabola and the line $y = 4x + 12$ intersect in the second quadrant at the point $(-2, 4)$.

To find the area of $R$, we integrate the function $f(x, y) = 1$ over $R_1$ and $R_2$; the area is

$$\iint_{R_1} 1 \, dA + \iint_{R_2} 1 \, dA$$

Decompose region.

$$= \int_{-2}^{0} \int_{x^2}^{4x+12} 1 \, dy \, dx + \int_{0}^{3} \int_{-x+12}^{x^2} 1 \, dy \, dx$$

Convert to iterated integrals.

$$= \int_{-2}^{0} (4x + 12 - x^2) \, dx + \int_{0}^{3} (-x + 12 - x^2) \, dx$$

Evaluate the inner integrals.

$$= \left( 2x^2 + 12x - \frac{x^3}{3} \right) \bigg|_{-2}^{0} + \left( \frac{x^2}{2} + 12x - \frac{x^3}{3} \right) \bigg|_{0}^{3}$$

Evaluate the outer integrals.

$$= \frac{40}{3} + \frac{45}{2} = \frac{215}{6}$$

Simplify.
**Question:** 10

Use a double integral to find the area of the region $R$ enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

**Solution**

The region $R$ may be treated equally well as type I (Figure 14.2.14a) or type II (Figure 14.2.14b).

Treating $R$ as type I yields

\[
\text{area of } R = \int_R \int dA = \int_0^4 \int_{y=\frac{1}{2}x^2}^{2x} dy \, dx = \int_0^4 \left[ y \right]_{y=\frac{1}{2}x^2}^{2x} \, dx = \int_0^4 \left( 2x - \frac{1}{2}x^2 \right) \, dx = \left[ x^2 - \frac{x^3}{6} \right]_0^4 = \frac{16}{3}
\]

Treating $R$ as type II yields

\[
\text{area of } R = \int_R \int dA = \int_0^8 \int_{y=x/2}^{\sqrt{2y}} dy \, dx = \int_0^8 \left[ x \right]_{x=\sqrt{2y}}^{\sqrt{y^2}} \, dy = \int_0^8 \left( \sqrt{2y} - \frac{1}{2}y \right) \, dy = \left[ \frac{2\sqrt{2}}{3} y^{3/2} - \frac{y^2}{4} \right]_0^8 = \frac{16}{3}
\]
**Question:** \(11\)

**Region bounded by two surfaces** Find the volume of the solid region bounded by the paraboloids \(z = x^2 + y^2\) and \(z = 8 - x^2 - y^2\) (Figure 13.23).

**Solution**

The upper surface bounding the solid is \(z = 8 - x^2 - y^2\) and the lower surface is \(z = x^2 + y^2\). The two surfaces intersect along a curve \(C\). Solving \(8 - x^2 - y^2 = x^2 + y^2\), we find that \(x^2 + y^2 = 4\). This circle of radius 2 is the projection of \(C\) onto the \(xy\)-plane (Figure 13.23); it is also the boundary of the region of integration

\[
R = \{(x, y); -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}, -2 \leq x \leq 2\}.
\]

Notice that \(R\) and the solid are symmetric about the \(x\)- and \(y\)-axes. Therefore, the volume of the entire solid is four times the volume over that part of \(R\) in the first quadrant. The volume of the solid is

\[
4 \int_{0}^{2} \int_{0}^{\sqrt{4-x^2}} \left( \frac{(8 - x^2 - y^2)}{g(x, y)} - \frac{(x^2 + y^2)}{f(x, y)} \right) dy \, dx
\]

\[
= 8 \int_{0}^{2} \int_{0}^{\sqrt{4-x^2}} (4 - x^2 - y^2) \, dy \, dx \quad \text{Simplify the integrand.}
\]
Use a double integral to find the volume of the indicated solid.

\[ z = \frac{y}{2} \]

\[ 0 \leq x \leq 4 \]
\[ 0 \leq y \leq 2 \]

1.

2. \( \int_0^4 \int_0^2 \frac{y}{2} \, dy \, dx = \int_0^4 \left[ \frac{y^2}{4} \right]_0^2 \, dx \)

3. \[ = \int_0^4 \, dx \]

4. \[ = 4 \]
Question : 13

Use a double integral to find the volume of the indicated solid.

1. 

2. \( \int_0^2 \int_0^y (4 - x - y) \, dx \, dy = \int_0^2 \left[ 4x - \frac{x^2}{2} - xy \right]_0^y \, dy 

3. 

4. 

5. 

6. 

= 4
Question : 14

Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0, 1] \times [0, 2]$ (Figure 14.1.6).

The volume is the double integral of $z = 4 - x - y$ over $R$.

$$\int_0^2 \int_0^1 (4 - x - y) \, dx \, dy \quad \text{or} \quad \int_0^1 \int_0^2 (4 - x - y) \, dy \, dx$$

Using the first of these, we obtain

$$V = \iint_R (4 - x - y) \, dA = \int_0^2 \int_0^1 (4 - x - y) \, dx \, dy$$

$$= \int_0^2 \left[ 4x - \frac{x^2}{2} - xy \right]_0^1 \, dy = \int_0^2 \left( \frac{7}{2} - y \right) \, dy$$

$$= \left[ \frac{7}{2}y - \frac{y^2}{2} \right]_0^2 = 5$$

You can check this result by evaluating the second integral.