MATH152 CALCULUS II TUTORIAL – 10
(25.12.2015)

Question 1:

Use polar coordinates to set up and evaluate the double integral \( \iint \! f(x, y) \, dA \).

\[ f(x, y) = x + y \quad \text{in} \quad R: \quad x^2 + y^2 \leq 4, \quad x \geq 0, \quad y \geq 0 \]

1. \[ \int_0^{\pi/2} \int_0^2 (x + y) \, dy \, dx = \int_0^{\pi/2} \int_0^2 (r \cos \theta + r \sin \theta) r \, dr \, d\theta \]
   \[ = \frac{\pi}{2} \int_0^2 (\cos \theta + \sin \theta) r^2 \, dr \, d\theta \]
2. \[ = \frac{8}{3} \int_0^{\pi/2} (\cos \theta + \sin \theta) \, d\theta \]
3. \[ = \left[ \frac{8}{3} (\sin \theta - \cos \theta) \right]_0^{\pi/2} \]
4. \[ = \frac{16}{3} \]

Question 3:

Evaluate the iterated integral by converting to polar coordinates.

\[ \int_0^2 \int_0^{\sqrt{4-x^2}} xy \, dy \, dx = \int_0^{\pi/2} \int_0^2 r^3 \cos \theta \sin \theta \, dr \, d\theta \]

1. \[ \int_0^{\pi/2} \int_0^2 xy \, dy \, dx = \int_0^{\pi/2} \int_0^2 r^3 \cos \theta \sin \theta \, dr \, d\theta \]
   \[ = 4 \int_0^{\pi/2} \cos^2 \theta \sin \theta \, d\theta \]
   \[ = \left[ -\frac{4}{3} \cos^3 \theta \right]_0^{\pi/2} \]
4. \[ = \frac{2}{3} \]

Question 4:

Evaluate the double integral \( \iint \! f(r, \theta) \, dA \), and sketch the region \( R \).

\[ \int_0^{\pi/2} \int_2^3 \sqrt{9 - r^2} \, r \, dr \, d\theta \]

1. \[ \int_0^{\pi/2} \int_2^3 \sqrt{9 - r^2} \, r \, dr \, d\theta = \int_0^{\pi/2} \left[ \frac{5 \sqrt{5} \theta}{3} \right]_0^{\pi/2} \]
2. \[ = \frac{5 \sqrt{5} \pi}{6} \]
4. Combine the sum of the two iterated integrals into a single iterated integral by converting to polar coordinates. Evaluate the resulting iterated integral.

\[ \int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{x^2 + y^2} \, dy \, dx + \int_2^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2 + y^2} \, dy \, dx \]

1. \[ = \sqrt{x^2 + y^2} \, dx \]
2. \[ = \sqrt{8 - x^2} \, dx \]
3. \[ = \frac{16 \sqrt{2}}{3} \]
4. \[ = \frac{4 \sqrt{2} \pi}{3} \]
Question 5:

Let $R$ be the annular region lying between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 5$. Evaluate the integral $\int_R (x^2 + y^2) \, dA$.

**Solution**

The polar boundaries are $1 \leq r \leq \sqrt{5}$ and $0 \leq \theta \leq 2\pi$, as shown in Figure 14.30. Furthermore, $x^2 = r \cos \theta$ and $y = r \sin \theta$. So, you have

\[
\int_R (x^2 + y^2) \, dA = \int_0^{2\pi} \int_1^{\sqrt{5}} (r^2 \cos^2 \theta + r^2 \sin^2 \theta) \cdot r \, dr \, d\theta \\
= \int_0^{2\pi} \int_1^{\sqrt{5}} (r^3 \cos^2 \theta + r^3 \sin^2 \theta) \, dr \, d\theta \\
= \int_0^{2\pi} \left( \frac{\sqrt{5}}{4} \cos^2 \theta + \frac{\sqrt{5}}{6} \sin^2 \theta \right) d\theta \\
= \int_0^{2\pi} \left( \frac{3 + 3 \cos 2\theta + 5\sqrt{5} - \frac{5}{3} \sin \theta}{2} \right) d\theta \\
= \left[ \theta + \frac{3 \sin 2\theta}{2} - \frac{5\sqrt{5} - 1}{3} \cos \theta \right]_0^{2\pi} \\
= 6\pi.
\]

Question 6:

Evaluate the triple integral.

\[
\int_0^1 \int_0^1 \int_0^{1-x} xy \, dz \, dy \, dx
\]

1. $\int_0^1 \int_0^1 \int_0^{1-x} xy \, dz \, dy \, dx = \int_0^1 \int_0^1 \left[ xy \cdot z \right]_0^{1-x} dy \, dx \\
= \int_0^1 \int_0^1 \left[ x \cdot (1-x) \cos y \right] dy \, dx \\
= \int_0^1 \left[ x \cdot (1-x) \sin y \right]_0^{\pi/2} dx \\
= \int_0^1 \left[ (1-x) \cos \theta \right] dx \\
= \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 \\
= 8 - \frac{64}{3} \\
= \frac{-40}{3}
\]

2. $\int_0^1 \int_0^1 \int_0^{1-x} xy \, dz \, dy \, dx
\]

3. $\int_0^1 \int_0^1 \int_0^{1-x} xy \, dz \, dy \, dx
\]

4. $\int_0^1 \int_0^1 \int_0^{1-x} xy \, dz \, dy \, dx
\]

Question 7:

Evaluate the triple integral

\[
\iiint_G 12xy^2 z^3 \, dV
\]

over the rectangular box $G$ defined by the inequalities

\[-1 \leq x \leq 2, \quad 0 \leq y \leq 3, \quad 0 \leq z \leq 2.\]

\[
\iiint_G 12xy^2 z^3 \, dV = \int_{-1}^2 \int_0^3 \int_0^2 12xy^2 z^3 \, dz \, dy \, dx \\
= \int_{-1}^2 \int_0^3 \left[ \frac{3xy^2 z^4}{4} \right]_0^2 \, dy \, dx \\
= \int_{-1}^2 \int_0^3 \frac{48xy^2}{4} \, dy \, dx \\
= \int_{-1}^2 \left[ \frac{16xy^3}{3} \right]_0^3 \, dx \\
= \int_{-1}^2 432x \, dx \\
= 216x^2 \bigg|_{-1}^2 = 648
\]

Question 8:

Use a triple integral to find the volume of the solid shown in the figure.

1. $\int_0^4 \int_0^4 \int_0^{4-x^2} dz \, dy \, dx = \int_0^4 (4-x^2)^3 \, dx \\
= \int_0^4 (16 - 8x^2 + x^4) \, dx \\
= \left[ \frac{16}{3}x^3 + \frac{1}{5}x^5 \right]_0^4 = 256 \frac{4}{15}
\]
**Question : 9**

Set up a triple integral for the volume of each solid region.

a. The region in the first octant bounded above by the cylinder $z = 1 - y^2$; between the vertical planes $x + y = 1$ and $x + y = 3$

b. The upper hemisphere given by $z = \sqrt{1 - x^2 - y^2}$

c. The region bounded below by the paraboloid $z = x^2 + y^2$ and above by the cylinder $z = 1 - y^2$.

a. In Figure 14.57, note that the solid is bounded below by the $xy$-plane ($z = 0$) above by the cylinder $z = 1 - y^2$. So,

$$0 \leq z \leq 1 - y^2.$$  

**Bounds for $z$**

Projecting the region onto the $xy$-plane produces a parallelogram. Because sides of the parallelogram are parallel to the $x$-axis, you have the following bounds for $y$:

$$1 - y \leq x \leq 3 - y$$ and $$0 \leq y \leq 1.$$  

So, the volume of the region is given by

$$V = \int_0^1 \int_{1-y}^{3-y} \int_0^{1-y^2} dz \, dx \, dy.$$  

![Figure 14.57](image)

b. For the upper hemisphere given by $z = \sqrt{1 - x^2 - y^2}$, you have

$$0 \leq z \leq \sqrt{1 - x^2 - y^2}.$$  

**Bounds for $z$**

In Figure 14.58, note that the projection of the hemisphere onto the $xy$-plane is the circle given by $x^2 + y^2 = 1$. If you can use either order $dx \, dy$ or $dy \, dx$. Choosing the first produces

$$-\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2} \quad \text{and} \quad -1 \leq y \leq 1.$$  

![Figure 14.58](image)

c. For the region bounded below by the paraboloid $z = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 6$, you have

$$x^2 + y^2 \leq z \leq \sqrt{6 - x^2 - y^2}.$$  

**Bounds for $z$**

The sphere and the paraboloid intersect at $z = 2$. Moreover, you can see in Figure 14.59 that the projection of the solid region onto the $xy$-plane is the circle given by $x^2 + y^2 = 2$. Using the order $dy \, dx$ produces

$$-\sqrt{2-x^2} \leq y \leq \sqrt{2-x^2} \quad \text{and} \quad -\sqrt{2} \leq x \leq \sqrt{2}.$$  

which implies that the volume of the region is given by

$$V = \int_0^\sqrt{2} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{-\sqrt{6-x^2-y^2}}^{\sqrt{6-x^2-y^2}} dz \, dy \, dx.$$  

![Figure 14.59](image)
Question: 10

Sketch the solid whose volume is given by the iterated integral and rewrite the integral using the indicated order of integration.

\[ \int_0^1 \int_y^1 \sqrt{1 - z^2} \, dz \, dx \, dy \]

Rewrite using the order \( dz \, dy \, dx \).

1.

2. Top cylinder: \( y^2 + z^2 = 1 \)
3. Side plane: \( x = y \)
4. \( \int_0^1 \int_y^1 \sqrt{1 - y^2} \, dz \, dy \, dx \)