CALCULUS I  TUTORIAL 12

Question 1

Write the first five terms of the sequence.

\( a_n = \sin \frac{n \pi}{2} \)

1. \( a_1 = \sin \frac{\pi}{2} = 1 \)
2. \( a_2 = \sin \pi = 0 \)
3. \( a_3 = \sin \frac{3\pi}{2} = -1 \)
4. \( a_4 = \sin 2\pi = 0 \)
5. \( a_5 = \sin \frac{5\pi}{2} = 1 \)

Question 2

Find the limit (if possible) of the sequence.

\( a_n = \frac{2n}{\sqrt{n^2 + 1}} \)

1. \( \lim_{n \to \infty} \frac{2n}{\sqrt{n^2 + 1}} = \lim_{n \to \infty} \frac{2}{\sqrt{1 + (1/n^2)}} = 2 \)
2. \( = \frac{2}{1} \)
3. \( = 2 \)

Question 3

Match the sequence with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

\( a_n = 4(0.5)^{n-1} \)

1. \( a_1 = 4 \)
2. \( a_2 = 2 \)
3. Decreases to 0
4. Matches (c).

Question 4

Determine the convergence or divergence of the sequence with the given nth term. If the sequence converges, find its limit.

\( a_n = \frac{3n^2 - n + 4}{2n^2 + 1} \)

1. \( \lim_{n \to \infty} \frac{3n^2 - n + 4}{2n^2 + 1} = \frac{3}{2} \)
2. Converges
Question 5

Verify that the infinite series diverges.

\[ \sum_{n=0}^{\infty} \left( \frac{3}{2} \right)^n \]

1. Geometric series
2. \( r = \frac{3}{2} > 1 \)
3. Diverges by Theorem 9.6

That is diverges by geometric series test

Question 6

Verify that the infinite series diverges.

\[ \sum_{n=0}^{\infty} 1000(1.055)^n \]

1. Geometric series
2. \( r = 1.055 > 1 \)
3. Diverges by Theorem 9.6

That is diverges by geometric series test

Question 7

Verify that the infinite series diverges.

\[ \sum_{n=1}^{\infty} \frac{n^2}{n^2 + 1} \]

1. \( \lim_{n \to \infty} \frac{n^2}{n^2 + 1} = 1 \)
2. \( \neq 0 \)
3. Diverges by Theorem 9.9

which is the divergence Theorem or nth term test.

Question 8

for the following geometric series

a)

Find the sum of the convergent series.

\[ 1 + 0.1 + 0.01 + 0.001 + \cdots \]

1. \[ \sum_{n=0}^{\infty} \left( \frac{1}{10} \right)^n \]
2. \[ = \frac{1}{1 - (1/10)} \]

b)

Find the sum of the convergent series.

\[ \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n \]

1. \[ \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n \]
2. \[ = \frac{2}{3} \]
Question 9

a)

Use Theorem 9.11 to determine the convergence or divergence of the $p$-series.

\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \]

1. \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \]

2. Divergent $p$-series with $p = \frac{1}{2} < 1$

b)

Use Theorem 9.11 to determine the convergence or divergence of the $p$-series.

\[ \sum_{n=1}^{\infty} \frac{1}{n^{1/3}} \]

1. Convergent $p$-series with $p = 1.04 > 1$

Question 10

Use the Limit Comparison Test to determine the convergence or divergence of the series.

\[ \sum_{n=1}^{\infty} \frac{2n^2 - 1}{3n^5 + 2n + 1} \]

1. \[ \lim_{n \to \infty} \frac{2n^2 - 1}{3n^5 + 2n + 1} = \lim_{n \to \infty} \frac{2n^5 - n^3}{3n^5 + 2n + 1} \]

2. \[ = \frac{2}{3} \]

3. Therefore,\[ \sum_{n=1}^{\infty} \frac{2n^2 - 1}{3n^5 + 2n + 1} \]

converges by a limit comparison with the convergent $p$-series

\[ \sum_{n=1}^{\infty} \frac{1}{n^3} \]

Question 11

Use the Limit Comparison Test to determine the convergence or divergence of the series.

\[ \sum_{n=1}^{\infty} \frac{2n^2 - 1}{3n^5 + 2n + 1} \]

1. \[ \lim_{n \to \infty} \frac{2n^2 - 1}{3n^5 + 2n + 1} = \lim_{n \to \infty} \frac{2n^5 - n^3}{3n^5 + 2n + 1} \]

2. \[ = \frac{2}{3} \]

3. Therefore,\[ \sum_{n=1}^{\infty} \frac{2n^2 - 1}{3n^5 + 2n + 1} \]

converges by a limit comparison with the convergent $p$-series

\[ \sum_{n=1}^{\infty} \frac{1}{n^3} \]

Question 12

Use the Root Test to determine the convergence or divergence of the series.

\[ \sum_{n=1}^{\infty} \left( \frac{r}{2n + 1} \right)^n \]

1. \[ \lim_{n \to \infty} \sqrt[n]{\left| r \right|^n} = \lim_{n \to \infty} \sqrt[n]{\left( \frac{r}{2n + 1} \right)^n} \]

2. \[ = \lim_{n \to \infty} \frac{n}{2n + 1} \]

3. \[ = \frac{1}{2} \]

4. Therefore, by the Root Test, the series converges.
Question 13

Use the Root Test to determine the convergence or divergence of the series.
\[
\sum_{n=1}^{\infty} (2\sqrt{n} + 1)^n
\]

1. \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \sqrt[n]{(2\sqrt{n} + 1)^n} \)
2. \( = \lim_{n \to \infty} (2\sqrt{n} + 1) \)
3. To find \( \lim \sqrt[n]{n} \), let \( y = \lim \sqrt[n]{n} \).
4. Then
   \( \ln y = \lim_{n \to \infty} \left( \ln \sqrt[n]{n} \right) \)
5. \( = \lim_{n \to \infty} \frac{1}{n} \ln n \)
6. \( = \lim_{n \to \infty} \frac{\ln n}{n} \)
7. \( = \lim_{n \to \infty} 1/n \)
8. \( = 0 \).
9. Thus, \( \ln y = 0 \),
10. so \( y = e^0 \)
11. \( = 1 \)
12. and \( \lim_{n \to \infty} (2\sqrt{n} + 1) = 2(1) + 1 \)
13. \( = 3 \).
14. Therefore, by the Root Test, the series diverges.

Question 15

Use the Ratio Test to determine the convergence or divergence of the series.
\[
\sum_{n=1}^{\infty} \left( \frac{3}{4} \right)^n
\]

1. \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n + 1)(3/4)^{n+1}}{n(3/4)^n} \right| \)
2. \( = \lim_{n \to \infty} \left| \frac{3(n + 1)}{4n} \right| \)
3. \( = \frac{3}{4} \)
4. Therefore, by the Ratio Test, the series converges.

Question 16

Use the Ratio Test to determine the convergence or divergence of the series.
\[
\sum_{n=1}^{\infty} \frac{n!}{n^3 n}
\]

1. \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n + 1)!}{(n + 1)^3 n!} \cdot n! \right| \)
2. \( = \lim_{n \to \infty} \frac{n}{3} \)
3. \( = \infty \)
4. Therefore, by the Ratio Test, the series diverges.
Question 16

a) 

Determine the convergence or divergence of the series.
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n - 1} \]

1. \[ a_{n+1} = \frac{1}{2(n + 1) - 1} \]

2. \[ \frac{1}{2n - 1} \]

3. \[ = a_n \]

4. \[ \lim_{n \to \infty} \frac{1}{2n - 1} = 0 \]

5. Converges by Theorem 9.14

which is the alternating series test

b) 

Determine the convergence or divergence of the series.
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n}n^2}{n^2 + 1} \]

1. \[ \lim_{n \to \infty} \frac{n^2}{n^2 + 1} = 1 \]

2. Thus, \[ \lim_{n \to \infty} a_n \neq 0. \]

3. Diverges by the nth-Term Test

Question 17

Determine the convergence or divergence of the series.
\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \]

1. \[ a_{n+1} = \frac{1}{(n + 1)!} \]

2. \[ < \frac{1}{n!} \]

3. \[ = a_n \]

4. \[ \lim_{n \to \infty} \frac{1}{n!} = 0 \]

5. Converges by Theorem 9.14

which is the alternating series test.