CALCULUS I TUTORIAL II

1.3 Limits at infinity

Q1)
Find the limit. (Hint: Treat the expression as a fraction whose denominator is 1, and rationalize the numerator.) Use a graphing utility to verify your result.

\[ \lim_{x \to -\infty} \left( x + \sqrt{x^2 + 3} \right) \]

1. \[ \lim_{x \to -\infty} \left( x + \sqrt{x^2 + 3} \right) = \lim_{x \to -\infty} \left[ x + \sqrt{x^2 + 3} \right] \cdot \frac{x - \sqrt{x^2 + 3}}{x - \sqrt{x^2 + 3}} \]

2. \[ = \lim_{x \to -\infty} \frac{-3}{x - \sqrt{x^2 + 3}} \]

3. \[ = 0 \]

Q2)

Find the limit. (Hint: Treat the expression as a fraction whose denominator is 1, and rationalize the numerator.) Use a graphing utility to verify your result.

\[ \lim_{x \to -\infty} (x - \sqrt{x^2 + x}) \]

1. \[ \lim_{x \to -\infty} (x - \sqrt{x^2 + x}) = \lim_{x \to -\infty} \left[ x - \sqrt{x^2 + x} \right] \cdot \frac{x + \sqrt{x^2 + x}}{x + \sqrt{x^2 + x}} \]

2. \[ = \lim_{x \to -\infty} \frac{-x}{x + \sqrt{x^2 + x}} \]

3. \[ = \lim_{x \to -\infty} \frac{-1}{1 + \sqrt{1 + (1/x)}} \]

4. \[ = \frac{1}{2} \]

Q3)

Find the limit.

\[ \lim_{x \to -\infty} \frac{2x^2}{3x^2 + 5} \]

1. \[ \lim_{x \to -\infty} \frac{2x^2}{3x^2 + 5} = \lim_{x \to -\infty} \frac{2}{3 + (5/x^2)} \]

2. \[ = \frac{2}{3} \]

Q4) Evaluate the following limits

a) \[ \lim_{x \to -\infty} \frac{x - 2}{x^2 + 2x + 1} \]

b) \[ \lim_{x \to -\infty} \frac{3x + 1}{2x - 5} \]

\[ \lim_{x \to -\infty} \frac{x - 2}{x^2 + 2x + 1} = \frac{1/2}{1 + 1/(1/x)} = 0 \]

b) \[ \lim_{x \to -\infty} \frac{3x + 1}{2x - 5} = \lim_{x \to -\infty} \frac{x \left( \frac{3 + 1}{x} \right)}{x \left( \frac{2 - 5}{x} \right)} = \frac{3}{2} \]

Q5)

a. Find the horizontal asymptote(s) of

\[ f(x) = \frac{x + x^3 + 4x^5}{1 + x^2 - x^3} \]

\[ \lim_{x \to \infty} x + x^3 + 4x^5 = \lim_{x \to \infty} x^5 \left( \frac{1/x^5 + 1/x^3 + 4}{1/x^5 + 1/x^3 - 1} \right) = -4 \]

also

\[ \lim_{x \to -\infty} x + x^3 + 4x^5 = \lim_{x \to -\infty} x^5 \left( \frac{1/x^5 + 1/x^3 + 4}{1/x^5 + 1/x^3 - 1} \right) = -4 \]

so \[ y = -4 \]

Is two sided horizontal asymptote.

b. Find the vertical asymptote(s) of

\[ f(x) = \frac{1 - x^2}{x - x^2} \]

\[ f(x) = \frac{1 - x^2}{x - x^2} \text{ is not defined at } x = 0 \text{ and } x = 1 \]
\[
\lim_{x \to a} \frac{1-x^2}{x-x^2} = \lim_{x \to a} \frac{(1-x)(1+x)}{x(1-x)} = \lim_{x \to a} \frac{1+x}{x} = -\infty
\]

\[
\lim_{x \to a} \frac{1-x^2}{x-x^2} = \lim_{x \to a} \frac{(1-x)(1+x)}{x(1-x)} = \lim_{x \to a} \frac{1+x}{x} = \infty
\]

so \( x = 0 \) is two sided vertical asymptote

\[
\lim_{x \to 1} \frac{1-x^2}{x-x^2} = \lim_{x \to 1} \frac{(1-x)(1+x)}{x(1-x)} = \lim_{x \to 1} \frac{1+x}{x} = 2
\]

therefore \( x = 1 \) is not a vertical asymptote.

### 1.5 Continuity

**Q6)**

Verify that the Intermediate Value Theorem applies to the indicated interval and find the value of \( c \) guaranteed by the theorem.

\[ f(x) = x^2 + x - 1, \quad [0, 5], \quad f(c) = 11 \]

1. \( f(x) = x^2 + x - 1 \)
2. \( f \) is continuous on \([0, 5]\).
3. \( f(0) = -1 \)
4. and \( f(5) = 29 \)
5. \(-1 < 11 < 29 \)
6. The Intermediate Value Theorem applies.
7. \( x^2 + x - 1 = 11 \)
8. \( x^2 + x = 12 = 0 \)
9. \( x + 10(x - 3) = 0 \)
10. \( x = -4 \)
11. or \( x = 3 \)
12. \( c = 3 \) (\( x = -4 \) is not in the interval.)
13. Thus, \( f(3) = 11 \).

**Q7)** Show that \( f(x) = x^x + 8x - 1 \) has a zero that is \( x^5 + 8x - 1 = 0 \) has a root in the interval \([0,1]\) by using Intermediate Value Theorem

\[ f(x) = x^x + 8x - 1 \] is continuous on \([0,1]\) and \( f(0) = -1, f(1) = 8 \). Since \( f \) is continuous on \([0,1]\) and \(-1 < 0 < 8 \) according to Intermediate Value Theorem there exist at least one number \( c \) in the interval \([0,1]\) such that \( f(c) = c^5 + 8c - 1 = 0 \) which the number \( c \) is the root of the given equation.

Find the constant \( a \), or the constants \( a \) and \( b \), such that the function is continuous on the entire real line.

\[ f(x) = \begin{cases} 2, & x \leq -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \geq 3 \end{cases} \]

1. Find \( a \) and \( b \) such that
   \[ \lim_{x \to -1^+} (ax + b) = -a + b = 2 \text{ and } \lim_{x \to -1^-} (ax + b) = 3a + b = -2. \]
2. \( a - b = -2 \)
3. \( (a) 3a + b = -2 \)
4. \( 4a = -4 \)
5. \( a = -1 \)
6. \( b = 2 + (-1) = 1 \)

**Q8)**

Find the \( x \)-values (if any) at which \( f \) is not continuous. Which of the discontinuities are removable?

\[ f(x) = \frac{x + 2}{x^2 - 3x - 10} \]

1. \( f(x) = \frac{x + 2}{(x + 2)(x - 5)} \) has a nonremovable discontinuity at \( x = 5 \)
2. since \( \lim_{x \to 5} f(x) \) does not exist
3. and has a removable discontinuity at \( x = -2 \)
4. since \( \lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{1}{x - 5} \)
5. \( = -\frac{1}{7} \)}
Q10)

Determine the intervals on which the function is continuous.

\[ f(x) = \begin{cases} \frac{3x^2 - x - 2}{x - 1}, & x \neq 1 \\ 0, & x = 1 \end{cases} \]

1. \( f(x) = \frac{3x^2 - x - 2}{x - 1} \)
2. \( \frac{(3x + 2)(x - 1)}{x - 1} = 3x + 2 \)
3. \( \lim_{x \to 1} f(x) = \lim_{x \to 1} (3x + 2) \)
4. \( = 5 \)
5. Removable discontinuity at \( x = 1 \)
6. Continuous on \( (-\infty, 1) \cup (1, \infty) \)

Q11)

Determine the value of \( c \) such that the function is continuous on the entire real line.

\[ f(x) = \begin{cases} x + 3, & x \leq 2 \\ cx + 6, & x > 2 \end{cases} \]

1. \( f(2) = 5 \)
2. Find \( c \) so that \( \lim_{x \to 2} (cx + 6) = 5 \).
3. \( c(2) + 6 = 5 \)
4. \( 2c = -1 \)
5. \( c = -\frac{1}{2} \)

Q12)

Find the \( x \)-values (if any) at which \( f \) is not continuous. Which of the discontinuities are removable?

\( f(x) = \begin{cases} \frac{1}{2}x + 1, & x \leq 2 \\ 3 - x, & x > 2 \end{cases} \)

1. \( f(x) = \frac{1}{2}x + 1 \) has a possible discontinuity at \( x = 2 \).
2. \( f(2) = \frac{1}{2}(2) + 1 = 2 \)
3. \( \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (\frac{1}{2}x + 1) = 2 \)
4. Therefore, \( f \) has a nonremovable discontinuity at \( x = 2 \).

This is jump type discontinuity.

1.6 Continuity of Trigonometric, Exponential and Inverse Functions

Q13)

Use the Squeeze Theorem to find \( \lim_{x \to \pi/2} f(x) \).

\[ c = 0, \ 4 - x^2 \leq f(x) \leq 4 + x^2 \]

1. \( \lim_{x \to \pi/2} (4 - x^2) \leq \lim_{x \to \pi/2} f(x) \leq \lim_{x \to \pi/2} (4 + x^2) \)
2. \( 4 \leq \lim_{x \to \pi/2} f(x) \leq 4 \)
3. Therefore, \( \lim_{x \to \pi/2} f(x) = 4 \).

Q14)

Find the limit of the trigonometric function.

\[ \lim_{x \to 0} \frac{\sin 2x}{2x} \]

1. \( \lim_{x \to 0} 2x = \sin 0 \)
2. \( = 1 \)

Q15)

Find the limit of the trigonometric function.

\[ \lim_{x \to \pi/2} \sin x \]

1. \( \lim_{x \to \pi/2} \sin x = \sin \frac{\pi}{2} \)
2. \( = 1 \)
Q16)

Use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.

\[
\lim_{\theta \to 0} \frac{\sin 3\theta}{\theta}
\]

1. \( f(\theta) = \frac{\sin 3\theta}{\theta} \)

2.

3. | \( \theta \) | -0.1 | -0.01 | -0.001 | 0 | 0.001 | 0.01 | 0.1 |
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<td>3</td>
<td>?</td>
<td>3</td>
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4. The limit appears to equal 3.

5. Analytically, \( \lim_{\theta \to 0} \frac{\sin 3\theta}{\theta} = 3 \lim_{\theta \to 0} \left( \frac{\sin 3\theta}{3\theta} \right) \)

6. = 3(1)

7. = 3.

Q17)

Use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.

\[
\lim_{x \to 0} \frac{\sin x^2}{x}
\]

1. \( f(x) = \frac{\sin x^2}{x} \)

2.

3. | \( x \) | -0.1 | -0.01 | -0.001 | 0 | 0.001 | 0.01 | 0.1 |
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<td>?</td>
<td>0.001</td>
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<td>0.00998</td>
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4. Analytically, \( \lim_{x \to 0} \frac{\sin x^2}{x} = \lim_{x \to 0} \left( \frac{\sin x^2}{x^2} \right) \)

5. = 0(1)

6. = 0.

Q18)

Determine the limit of the trigonometric function (if it exists).

\[
\lim_{h \to 0} \frac{(1 - \cos h)^2}{h}
\]

1. \( \lim_{h \to 0} \frac{(1 - \cos h)^2}{h} = \lim_{h \to 0} \left[ \frac{1 - \cos h}{h} \right] \left( 1 - \cos h \right) \)

2. = (0)(0)

3. = 0

Q19)

Determine the limit of the trigonometric function (if it exists).

\[
\lim_{x \to 0} \frac{\sin x (1 - \cos x)}{2x^2}
\]

1. \( \lim_{x \to 0} \frac{\sin x (1 - \cos x)}{2x^2} = \lim_{x \to 0} \left[ \frac{1}{2} \cdot \frac{\sin x}{x} \cdot \frac{1 - \cos x}{x} \right] \)

2. = \( \frac{1}{2}(1)(0) \)

3. = 0