CALCULUS I  TUTORIAL II

1.3 Limits at infinity

Q1)

Find the limit. (Hint: Treat the expression as a fraction whose denominator is 1, and rationalize the numerator.) Use a graphing utility to verify your result.

\[ \lim_{x \to -\infty} \frac{x + \sqrt{x^2 + 3}}{x} = \lim_{x \to \infty} \frac{x + \sqrt{x^2 + 3}}{x - \sqrt{x^2 + 3}} \]

1. \[ \lim_{x \to -\infty} \frac{x + \sqrt{x^2 + 3}}{x} = \lim_{x \to \infty} \frac{x - \sqrt{x^2 + 3}}{x} = -3 \]

2. \[ \lim_{x \to -\infty} \frac{x + \sqrt{x^2 + 3}}{x} = 0 \]

Q2)

Find the limit. (Hint: Treat the expression as a fraction whose denominator is 1, and rationalize the numerator.) Use a graphing utility to verify your result.

\[ \lim_{x \to \infty} (x - \sqrt{x^2 + x}) \]

1. \[ \lim_{x \to \infty} (x - \sqrt{x^2 + x}) = \lim_{x \to \infty} \left( \frac{x - \sqrt{x^2 + x}}{x + \sqrt{x^2 + x}} \right) = -1 \]

2. \[ \lim_{x \to \infty} (x - \sqrt{x^2 + x}) = \lim_{x \to \infty} \frac{-1}{x + \sqrt{x^2 + x}} = 0 \]

3. \[ \lim_{x \to \infty} (x - \sqrt{x^2 + x}) = \lim_{x \to \infty} \frac{-1}{1 + \sqrt{1 + \frac{1}{x^2}}} = 0 \]

4. \[ \lim_{x \to \infty} (x - \sqrt{x^2 + x}) = \frac{1}{2} \]

Q4) Evaluate the following limits

a) \[ \lim_{x \to \infty} \frac{x - 2}{x^2 + 2x + 1} = \frac{x^2}{x^2 + 2x + 1} = \frac{1}{1 + \frac{2}{x^2} + \frac{1}{x^2}} = 0 \]

b) \[ \lim_{x \to \infty} \frac{3x + 1}{2x - 5} = \lim_{x \to \infty} \frac{x \left( 3 + \frac{1}{x} \right)}{2x - 5} = \frac{3}{2} \]

1.5 Continuity

Q5)

Verify that the Intermediate Value Theorem applies to the indicated interval and find the value of \( c \) guaranteed by the theorem.

\( f(x) = x^2 + x - 1, \ [0, 5] \)

1. \( f(x) = x^2 + x - 1 \)

2. \( f \) is continuous on \([0, 5]\).

3. \( f(0) = -1 \)

4. \( f(5) = 29 \)

5. \(-1 < 11 < 29 \)

6. The Intermediate Value Theorem applies.

7. \( x^2 + x - 1 = 11 \)

8. \( x^2 + x - 12 = 0 \)

9. \( (x + 4)(x - 3) = 0 \)

10. \( x = 4 \)

11. or \( x = 3 \)

12. \( c = 3 \) (\( x = -4 \) is not in the interval.)

13. Thus, \( f(3) = 11 \).
Q6) Find the constant $a$, or the constants $a$ and $b$, such that the function is continuous on the entire real line.

$$f(x) = \begin{cases} 
2, & x \leq -1 \\
ax + b, & -1 < x < 3 \\
-2, & x \geq 3
\end{cases}$$

1. Find $a$ and $b$ such that
   $$\lim_{x \to -1^-} (ax + b) = -a + b = 2$$
   and
   $$\lim_{x \to -1^+} (ax + b) = 3a + b = -2.$$ 

2. $a - b = -2$

3. $(+) 3a + b = -2$

4. $4a = -4$

5. $a = -1$

6. $b = 2 + (-1) = 1$

Q8) Determine the intervals on which the function is continuous.

$$f(x) = \begin{cases} 
\frac{3x^2 - x - 2}{x - 1}, & x \neq 1 \\
0, & x = 1
\end{cases}$$

1. $f(x) = \frac{3x^2 - x - 2}{x - 1}$

2. $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} (3x + 2)$

3. $\lim_{x \to 1} f(x) = 5$

4. Removable discontinuity at $x = 1$

5. Continuous on $(-\infty, 1) \cup (1, \infty)$

Q9) Determine the value of $c$ such that the function is continuous on the entire real line.

$$f(x) = \begin{cases} 
x + 3, & x \leq 2 \\
cx + 6, & x > 2
\end{cases}$$

1. $f(2) = 5$

2. Find $c$ so that $\lim_{x \to 2^-} (cx + 6) = 5$

3. $c(2) + 6 = 5$

4. $2c = -1$

5. $c = -\frac{1}{2}$

Q10) Find the $x$-values (if any) at which $f$ is not continuous. Which of the discontinuities are removable?

$$f(x) = \begin{cases} 
\frac{1}{3}x + 1, & x \leq 2 \\
3 - x, & x > 2
\end{cases}$$

1. $f(x) = \frac{1}{3}x + 1$, $x \leq 2$ has a possible discontinuity at $x = 2$.

2. $f(2) = \frac{1}{3}(2) + 1 = 2$

3. $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \left(\frac{1}{3}x + 1\right) = 2$

4. $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3 - x) = 1$

5. $\lim_{x \to 2} f(x)$ does not exist.

4. Therefore, $f$ has a nonremovable discontinuity at $x = 2$. 
1.6 Continuity of Trigonometric, Exponential and Inverse Functions

Q11)

Use the Squeeze Theorem to find \( \lim_{x \to c} f(x) \).

\[ c = 0, \quad 4 - x^2 \leq f(x) \leq 4 + x^2 \]

1. \( \lim_{x \to 0} (4 - x^2) \leq \lim_{x \to 0} f(x) \leq \lim_{x \to 0} (4 + x^2) \)
2. \( 4 \leq \lim_{x \to 0} f(x) \leq 4 \)
3. Therefore, \( \lim_{x \to 0} f(x) = 4 \).

Q12)

Find the limit of the trigonometric function.

\[ \lim_{x \to 0} \sec 2x \]

1. \( \lim_{x \to 0} \sec 2x = \sec 0 \)
2. \( = 1 \)

Q13)

Find the limit of the trigonometric function.

\[ \lim_{x \to \pi/2} \sin x \]

1. \( \lim_{x \to \pi/2} \sin x = \sin \frac{\pi}{2} \)
2. \( = 1 \)

Q14)

Use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.

\[ \lim_{x \to 0} \frac{\sin 3x}{x} \]

1. \( f(x) = \frac{\sin 3x}{x} \)
2. 
3. | \( t \) | \(-0.1\) | \(-0.01\) | \(-0.001\) | \(0\) | \(0.001\) | \(0.01\) | \(0.1\) |
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<td>( f(t) )</td>
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<td>3</td>
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4. The limit appears to equal 3.
5. Analytically, \( \lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} \left( \frac{\sin 3x}{3x} \right) \)
6. \( = 3(1) \)
7. \( = 3 \).

Q15)

Use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.

\[ \lim_{x \to 0} \frac{\sin x^2}{x} \]

1. \( f(x) = \frac{\sin x^2}{x} \)
2. 
3. | \( x \) | \(-0.1\) | \(-0.01\) | \(-0.001\) | \(0\) | \(0.001\) | \(0.01\) | \(0.1\) |
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4. Analytically, \( \lim_{x \to 0} \frac{\sin x^2}{x} = \lim_{x \to 0} \left( \frac{\sin x^2}{x^2} \right) \)
5. \( = 0(1) \)
6. \( = 0 \).
Q16)

Determine the limit of the trigonometric function (if it exists).

\[ \lim_{{h \to 0}} \frac{{(1 - \cos h)^2}}{h} \]

1. \[ \lim_{{h \to 0}} \frac{{(1 - \cos h)^2}}{h} = \lim_{{h \to 0}} \left[ \frac{{1 - \cos h}}{h} (1 - \cos h) \right] \]

2. \[ = (0)(0) \]

3. \[ = 0 \]

Q17)

Determine the limit of the trigonometric function (if it exists).

\[ \lim_{{x \to 0}} \frac{{\sin x (1 - \cos x)}}{2x^2} \]

1. \[ \lim_{{x \to 0}} \frac{{\sin x (1 - \cos x)}}{2x^2} = \lim_{{x \to 0}} \left[ \frac{1}{2} \frac{\sin x}{x} \frac{1 - \cos x}{x} \right] \]

2. \[ = \frac{1}{2}(1)(0) \]

3. \[ = 0 \]