CALCULUS I TUTORIAL 7

Absolute Maxima and Minima

Question 1

Locate the absolute extrema of the function on the closed interval.
\( y = 3x^{2/3} - 2x, [-1, 1] \)
1. \( y' = 2x^{1/3} - 2 \)
2. \( \frac{2(1 - \sqrt[3]{x})}{\sqrt[3]{x}} \)
3. Left endpoint: \((-1.5)\) Maximum
4. Critical number: \((0, 0)\) Minimum
5. Right endpoint: \((1, 1)\)

Note that at \((0,0)\) the derivative is not defined but the function is defined so this point is a singular point.

Question 2

Locate the absolute extrema of the function on the closed interval
\( f(x) = x^3 - \frac{3}{2}x^2, [-1, 2] \)
1. \( f'(x) = 3x^2 - 3x \)
2. \( = 3x(x - 1) \)
3. Left endpoint: \((-1, -\frac{3}{2})\) Minimum
4. Right endpoint: \((2, 2)\) Maximum
5. Critical number: \((0, 0)\)
6. Critical number: \((1, -\frac{1}{2})\)

Question 3

Locate the absolute extrema of the function on the closed interval.
\( g(t) = \frac{t^2}{t^2 + 3}, [-1, 1] \)
1. \( g'(t) = \frac{6t}{(t^2 + 3)^2} \)
2. Left endpoint: \((-1, \frac{1}{4})\) Maximum
3. Critical number: \((0, 0)\) Minimum
4. Right endpoint: \((1, \frac{1}{4})\) Maximum

Question 4

Locate the absolute extrema of the function on the closed interval.
\( y = \frac{4}{x} + \tan\left(\frac{\pi x}{8}\right), [1, 2] \)
1. \( y' = -\frac{4}{x^2} + \frac{\pi}{8} \sec^2 \frac{\pi x}{8} = 0 \)
2. \( \frac{\pi}{8} \sec^2 \frac{\pi x}{8} = \frac{4}{x^2} \)
3. On the interval \([1, 2]\), this equation has no solutions.
4. Thus, there are no critical numbers.
5. Left endpoint: \((1, \sqrt{2} + 3) = (1, 4.4442)\) Maximum
6. Right endpoint: \((2, 3)\) Minimum
**Question 5**

Sketch the graph of the function. Then locate the absolute extrema of the function over the given interval.

\[ f(x) = \begin{cases} 2x + 2, & 0 \leq x \leq 1 \\ \frac{2}{4x^2}, & 1 < x \leq 3 \end{cases}, \quad [0, 3] \]

1. Left endpoint: (0, 2) Minimum
2. Right endpoint: (3, 36) Maximum

**Roll’s Theorem ; Mean Value Theorem**

**Question 6**

Explain why Rolle’s Theorem does not apply to the function even though there exist a and b such that \( f(a) = f(b) \).

\[ f(x) = 1 - |x - 1| \]

1. Rolle’s Theorem does not apply to \( f(x) = 1 - |x - 1| \) over \([0, 2]\) since \( f \) is not differentiable at \( x = 1 \).

**Question 7**

Consider the graph of the function \( f(x) = x^2 + 1 \).
(a) Find the equation of the secant line joining the points \((-1, 2)\) and \((2, 5)\).
(b) Use the Mean Value Theorem to determine a point \( c \) in the interval \((-1, 2)\) such that the tangent line at \( c \) is parallel to the secant line.
(c) Find the equation of the tangent line through \( c \).
(d) Then use a graphing utility to graph \( f \), the secant line, and the tangent line.

1. (a) Slope \( \frac{5 - 2}{2 + 1} = \frac{3}{3} = 1 \)
2. Secant line: \( y - 2 = 1(x + 1) \)
3. \( y = x + 3 \)
4. (b) \( f'(x) = 2x \)
5. \( f'(\frac{1}{2}) = 1 \)
6. \( f(\frac{1}{2}) = \frac{5}{4} \)
7. \( c = \frac{1}{2} \)
8. Tangent line: \( y - \frac{5}{4} = 1(x - \frac{1}{2}) \)
9. \( y = x + \frac{3}{4} \)

12. (d)
Question 8

Determine whether the Mean Value Theorem can be applied to \( f \) on the closed interval \([a, b]\). If the Mean Value Theorem can be applied, find all values of \( c \) in the open interval \((a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

1. \( f(x) = \sqrt{2 - x}, \quad [-7, 2] \)
2. \( f(x) = \sqrt{2 - x} \) is continuous on \([-7, 2]\)
3. \( \frac{f(2) - f(-7)}{2 - (-7)} = \frac{0 - 3}{9} \)
4. \( \frac{-1}{3} = \frac{-1}{2\sqrt{2} - x} \)
5. \( x = \frac{1}{3} \)
6. \( 2\sqrt{2} - x = 3 \)
7. \( \sqrt{2} - x = \frac{3}{2} \)
8. \( 2 - x = \frac{9}{4} \)
9. \( x = -\frac{1}{4} \)
10. \( c = -\frac{1}{4} \)

Question 9

Use a graphing utility to graph the function on the closed interval \([a, b]\). Determine whether Rolle’s Theorem can be applied to \( f \) on the interval and, if so, find all values of \( c \) in the open interval \((a, b)\) such that \( f'(c) = 0 \).

\( f(x) = 4x - \tan \pi x, \quad \left[-\frac{1}{4}, \frac{1}{4}\right] \)

1. \[\text{Graph} \]
2. \( f\left(-\frac{1}{4}\right) = f\left(\frac{1}{4}\right) \)
3. \( = 0 \)
4. \( f \) is continuous on \([-1/4, 1/4]\).
5. \( f \) is differentiable on \((-1/4, 1/4)\).
6. Rolle’s Theorem applies.
7. \( f'(x) = 4 - \pi \sec^2 \pi x \)
8. \( = 0 \)
9. \( \sec^2 \pi x = \frac{4}{\pi} \)
10. \( \sec \pi x = \pm \frac{2}{\sqrt{\pi}} \)
11. \( x = \pm \frac{1}{\pi} \ \text{arcsec} \frac{2}{\sqrt{\pi}} \)
12. \( = \pm \frac{1}{\pi} \ \text{arccos} \frac{\sqrt{\pi}}{2} \)
13. \( = \pm 0.1533 \ \text{radian} \)
14. \( c\)-values: \( \pm 0.1533 \ \text{radian} \)
Question 10

Use a graphing utility to (a) graph the function \( f \) on the given interval, (b) find and graph the secant line through points on the graph of \( f \) at the endpoints of the given interval, and (c) find and graph any tangent lines to the graph of \( f \) that are parallel to the secant line.

\[
f(x) = \frac{x}{x + 1} + 1 - \frac{1}{2} \cdot 2
\]

1. (a)

2. (b) Secant line:

3. \[\text{Slope} = \frac{f(2) - f(-1/2)}{2 - (-1/2)}\]

4. \[= \frac{2/3 - (-1)}{5/2}\]

5. \[= \frac{2}{3}\]

6. \[y - \frac{2}{3} = \frac{2}{3}(x - 2)\]

7. \[3y - 2 = 2x - 4\]

8. \[3y - 2x - 2 = 0\]

9. (c) \[f'(x) = \frac{1}{(x + 1)^2}\]

10. \[= \frac{2}{3}\]

11. \[(x + 1)^2 = \frac{2}{3}\]

12. \[x = -1 \pm \sqrt{ \frac{2}{3} }\]

13. \[= -1 \pm \sqrt{ \frac{6}{3} }\]

14. In the interval \([-1/2, 2] \), \( c = -1 + (\sqrt{6}/2) \).

15. \[f(c) = \frac{-1 + (\sqrt{6}/2)}{[1 - (\sqrt{6}/2)] + 1}\]

16. \[= -2 + \sqrt{6}\]

17. \[= \frac{2}{\sqrt{6}} + 1\]

18. Tangent line: \[y - 1 + \frac{2}{\sqrt{6}} = \frac{2}{3}(x - \frac{\sqrt{6}}{3} + 1)\]

19. \[y - 1 + \frac{\sqrt{6}}{3} = \frac{2}{3}x - \frac{\sqrt{6}}{3} + \frac{2}{3}\]

20. \[3y - 2x - 5 + 2\sqrt{6} = 0\]

Question 11

Determine the values \( a, b, \) and \( c \) such that the function \( f \) satisfies the hypotheses of the Mean Value Theorem on the interval \([0, 3]\).

\[
f(x) = \begin{cases} 
1, & x = 0 \\
ax + b, & 0 < x \leq 1 \\
x^2 + 4x + c, & 1 < x \leq 3
\end{cases}
\]

1. \( f \) is continuous at \( x = 0; 1 = b \)

2. \( f \) continuous at \( x = 1; a + 1 = 5 + c \)

3. \( f \) differentiable at \( x = 1; a = 2 + 4 = 6 \)

4. Hence, \( c = 2 \).

5. \[f(x) = \begin{cases} 
1, & x = 0 \\
6x + 1, & 0 < x \leq 1 \\
x^2 + 4x + 2, & 1 < x \leq 3
\end{cases}
\]

6. \[= 6x + 1, \quad 0 \leq x \leq 1 \]

\[= x^2 + 4x + 2, \quad 1 < x \leq 3\]

Question 12

Determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

The Mean Value Theorem can be applied to \( f(x) = 1/x \) on the interval \([-1, 1]\).

1. False

2. \( f(x) = 1/x \) has a discontinuity at \( x = 0 \).

Determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

If the graph of a polynomial function has three \( x \)-intercepts, then it must have at least two points at which its tangent line is horizontal.

1. True

2. A polynomial is continuous and differentiable everywhere.