CALCULUS I TUTORIAL 6

Question 1

Find all relative extrema. Use the Second Derivative Test where applicable.
\[ f(x) = x^3 - 4x^2 + 2 \]
1. \[ f'(x) = 3x^2 - 12x \]
2. \[ f''(x) = 6x - 12 \]
3. \[ f''(x) = 6(x - 2) \]
4. Critical numbers: \( x = 0, x = 3 \)
5. However, \( f''(0) = 0 \), so we must use the First Derivative Test.
6. \( f'(x) < 0 \) on the intervals \( (-\infty, 0) \) and \( (0, 3) \); hence, \( (0, 2) \) is not an extremum.
7. \( f''(3) > 0 \) so \( (3, -25) \) is a relative minimum.

Question 2

Find all relative extrema. Use the Second Derivative Test where applicable.
\[ f(x) = x + \frac{4}{x} \]
1. \[ f'(x) = 1 - \frac{4}{x^2} \]
2. \[ f''(x) = \frac{8}{x^3} \]
3. Critical numbers: \( x = \pm 2 \)
4. Therefore, \( (2, -4) \) is a relative maximum.
5. \( f''(-2) < 0 \)
6. Therefore, \( (-2, -4) \) is a relative minimum.
7. \( f''(2) > 0 \)
8. Therefore, \( (2, 4) \) is a relative minimum.

Question 3

Find the points of inflection and discuss the concavity of the graph of the function.
\[ f(x) = x^3 - 6x^2 + 12x \]
1. \[ f'(x) = 3x^2 - 12x + 12 \]
2. \[ f''(x) = 6(x - 2) \]
3. \[ = 0 \text{ when } x = 2 \]
4. The concavity changes at \( x = 2 \).
5. \( (2, 8) \) is a point of inflection.
6. Concave upward: \( (2, \infty) \)
7. Concave downward: \( (-\infty, 2) \)

Question 4

Find the points of inflection and discuss the concavity of the graph of the function.
\[ f(x) = \frac{1}{4} x^4 - 2x^2 \]
1. \[ f'(x) = x^3 - 4x \]
2. \[ f''(x) = 3x^2 - 4 \]
3. \[ = 0 \text{ when } x = \pm \frac{2}{\sqrt{3}} \]
4. Test interval:

<table>
<thead>
<tr>
<th>Test interval:</th>
<th>(-\infty &lt; x &lt; -\frac{2}{\sqrt{3}})</th>
<th>(-\frac{2}{\sqrt{3}} &lt; x &lt; \frac{2}{\sqrt{3}})</th>
<th>(\frac{2}{\sqrt{3}} &lt; x &lt; \infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sign of (f''(x)):</td>
<td>(f''(x) &gt; 0)</td>
<td>(f''(x) &lt; 0)</td>
<td>(f''(x) &gt; 0)</td>
</tr>
<tr>
<td>Conclusion:</td>
<td>Concave upward</td>
<td>Concave downward</td>
<td>Concave upward</td>
</tr>
</tbody>
</table>

5. Points of inflection: \( \left( \pm \frac{2}{\sqrt{3}}, \frac{20}{9} \right) \)
Question 5

Find the points of inflection and discuss the concavity of the graph of the function.

\[ f(x) = x(x-4)^3 \]

1. \[ f'(x) = 3(x-4)^2 + (x-4)^3 \]
2. \[ = (x-4)^2(4x-4) \]
3. \[ f''(x) = 4(x-1)[2(x-4)] + 4(x-4)^2 \]
4. \[ = 4(x-4)[2(x-1) + (x-4)] \]
5. \[ = 4(x-4)(3x-6) \]
6. \[ = 12(x-4)(x-2) \]
7. \[ f'''(x) = 12(x-4)(x-2) \]
8. \[ = 0 \text{ when } x = 2, 4. \]

9. Test interval: \(-\infty < x < 2 \quad 2 < x < 4 \quad 4 < x < \infty\)

<table>
<thead>
<tr>
<th>Sign of (f''(x))</th>
<th>Conclusion</th>
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<tbody>
<tr>
<td>(f''(x) &gt; 0)</td>
<td>Concave upward</td>
</tr>
<tr>
<td>(f''(x) &lt; 0)</td>
<td>Concave down</td>
</tr>
<tr>
<td>(f''(x) &gt; 0)</td>
<td>Concave upward</td>
</tr>
</tbody>
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10. Points of inflection: \((2, -16), (4, 0)\)

Question 6

Analyze and sketch a graph of the function. Label any intercepts, relative extrema, points of inflection, and asymptotes. Use a graphing utility to verify your results.

\[ y = x^3 - 3x^2 + 3 \]

1. \[ y' = 3x^2 - 6x \]
2. \[ = 3x(x-2) \]
3. \[ = 0 \text{ when } x = 0, x = 2. \]
4. \[ y'' = 6x - 6 \]
5. \[ = 6(x-1) \]
6. \[ = 0 \text{ when } x = 1. \]

<table>
<thead>
<tr>
<th>(y)</th>
<th>(y')</th>
<th>(y'')</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\infty &lt; x &lt; 0)</td>
<td>+</td>
<td>-</td>
<td>Increasing, concave down</td>
</tr>
<tr>
<td>(x = 0)</td>
<td>3</td>
<td>0</td>
<td>Relative maximum</td>
</tr>
<tr>
<td>(0 &lt; x &lt; 1)</td>
<td>-</td>
<td>-</td>
<td>Decreasing, concave down</td>
</tr>
<tr>
<td>(x = 1)</td>
<td>1</td>
<td>0</td>
<td>Relative minimum</td>
</tr>
<tr>
<td>(1 &lt; x &lt; \infty)</td>
<td>-</td>
<td>+</td>
<td>Decreasing, concave up</td>
</tr>
<tr>
<td>(x = 2)</td>
<td>-1</td>
<td>0</td>
<td>Relative maximum</td>
</tr>
<tr>
<td>(2 &lt; x &lt; \infty)</td>
<td>+</td>
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8. Diagram with points labeled: (0, 3), (2, 5), (3, 1)
Question 7

Analyze and sketch a graph of the function. Label any intercepts, relative extrema, points of inflection, and asymptotes. Use a graphing utility to verify your results.

\[ f(x) = \frac{x^2 + 1}{x} \]

1. \[ f'(x) = x + \frac{1}{x} \]
2. \[ f'(x) = 1 - \frac{1}{x^2} \]
3. \[ = 0 \text{ when } x = \pm 1. \]
4. \[ f''(x) = \frac{2}{x^3} \neq 0 \]
5. Relative maximum: \((-1, -2)\)
6. Relative minimum: \((1, 2)\)
7. Vertical asymptote: \(x = 0\)
8. Slant asymptote: \(y = x\)

\[ y = \frac{x^2}{x^2 + 3} \]

1. \[ y' = \frac{6x}{(x^2 + 3)^2} \]
2. \[ = 0 \text{ when } x = 0. \]
3. \[ y'' = \frac{18(1 - x^2)}{(x^2 + 3)^3} \]
4. \[ = 0 \text{ when } x = \pm 1. \]
5. Horizontal asymptote: \(y = 1\)

<table>
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<tr>
<td>(-\infty &lt; x &lt; -1)</td>
<td>(-)</td>
<td>(-)</td>
<td>Decreasing, concave down</td>
</tr>
<tr>
<td>(x = -1)</td>
<td>(-\frac{1}{2})</td>
<td>(-)</td>
<td>Point of inflection</td>
</tr>
<tr>
<td>(-1 &lt; x &lt; 0)</td>
<td>(-)</td>
<td>(+)</td>
<td>Decreasing, concave up</td>
</tr>
<tr>
<td>(x = 0)</td>
<td>(0)</td>
<td>(0)</td>
<td>Relative minimum</td>
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<td>(0 &lt; x &lt; 1)</td>
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\[ y = \frac{1}{x} \]

7. \[ y = \frac{1}{x} \]

\[ (-1, 1) \quad (0, 0) \quad (1, 1) \]