Analysis of Functions

Question 1. For the function \( f(x) = 2x^3 + 3x^2 - 12x \)

a) Find the critical points (if exist) and their nature

\[ f'(x) = 6x^2 + 6x - 12 \]

let \( f'(x) = 0 \)

\[ \Rightarrow 6(x^2 + x - 2) = 0 \]

\[ \Rightarrow 6(x + 2)(x - 1) = 0 \]

\[ \Rightarrow x = 1, x = -2 \] are critical values

and \( f'(x) \) is defined for all \( x \in \mathbb{R} \)

\[
\begin{array}{|c|c|c|c|}
\hline
& -\infty < x < -2 & -2 < x < 1 & 1 < x < \infty \\
\hline
f''(x) = 6x^2 + 6x - 12 & + & - & + \\
\hline
f(x) = 2x^3 + 3x^2 - 12x & Inc. & Dec. & Inc. \\
\hline
\end{array}
\]

Local (Relative) maximum occurs at \( x = -2 \) and \( f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) = 20 \)

Local (Relative) minimum occurs at \( x = 1 \) and \( f(1) = 2(1)^3 + 3(1)^2 - 12(1) = -7 \)

b) Write down the increasing and decreasing intervals

\( f(x) \) increases on \( (-\infty, -2) \cup (1, \infty) \) and decreases on \( (-2, 1) \)

c) Find the inflection points (if exist)

\[ f''(x) = 12x + 6 \] let \( f''(x) = 0 \Rightarrow 12x = -6 \Rightarrow x = -\frac{1}{2} \)

\[ f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 + 3\left(-\frac{1}{2}\right)^2 - 12\left(-\frac{1}{2}\right) \]

\[ = \frac{13}{2} \Rightarrow \left(-\frac{1}{2}, \frac{13}{2}\right) \]

is an inflection point since the concavity is opposite before and after this point.
d) Write down the concave up and concave down intervals

\[ f'(x) \text{ is concave up on } \left( -\frac{1}{2}, \infty \right) \text{ and concave down on } \left( -\infty, -\frac{1}{2} \right) \text{ as shown in the figure.} \]

<table>
<thead>
<tr>
<th>( f''(x) = 12x + 6 )</th>
<th>(-\infty &lt; x &lt; -\frac{1}{2})</th>
<th>(-\frac{1}{2} &lt; x &lt; \infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = 2x^3 + 3x^2 - 12x )</td>
<td>( \cap ) (concave down)</td>
<td>( \cup ) (concave up)</td>
</tr>
</tbody>
</table>

**Question 2**

Find all relative extrema. Use the Second Derivative Test where applicable.

\[ f(x) = x^4 - 4x^3 + 2 \]

1. \( f'(x) = 4x^3 - 12x^2 \)
2. \( f''(x) = 4x^2(x - 3) \)
3. \( f''(x) = 12x^2 - 24x \)
4. \( f''(x) = 12x(x - 2) \)
5. Critical numbers: \( x = 0, x = 3 \)
6. However, \( f''(0) = 0 \), so we must use the First Derivative Test.
7. \( f'(x) < 0 \) on the intervals \((-\infty, 0)\) and \((0, 3)\); hence, \((0, 2)\) is not an extremum.
8. \( f''(3) > 0 \) so \((3, -25)\) is a relative minimum.

**Question 3**

Find all relative extrema. Use the Second Derivative Test where applicable.

\[ f(x) = x + \frac{4}{x} \]

1. \( f'(x) = 1 - \frac{4}{x^2} \)
2. \( f''(x) = \frac{x^2 - 4}{x^2} \)
3. \( f''(x) = \frac{8}{x^3} \)
4. Critical numbers: \( x = \pm 2 \)
5. \( f''(-2) < 0 \)
6. Therefore, \((-2, -4)\) is a relative maximum.
7. \( f''(2) > 0 \)
8. Therefore, \((2, 4)\) is a relative minimum.
Question 4

Find the points of inflection and discuss the concavity of the graph of the function.

\( f(x) = x^3 - 6x^2 + 12x \)

1. \( f'(x) = 3x^2 - 12x + 12 \)
2. \( f''(x) = 6(x - 2) \)
3. \( = 0 \) when \( x = 2 \).
4. The concavity changes at \( x = 2 \).
5. \((2, 8)\) is a point of inflection.
6. Concave upward: \((2, \infty)\)
7. Concave downward: \((-\infty, 2)\)

Question 5

Find the points of inflection and discuss the concavity of the graph of the function.

\( f(x) = \frac{1}{4}x^4 - 2x^2 \)

1. \( f'(x) = x^3 - 4x \)
2. \( f''(x) = 3x^2 - 4 \)
3. \( = 0 \) when \( x = \pm \frac{2}{\sqrt{3}} \).
4. Test interval: \(-\infty < x < -\frac{2}{\sqrt{3}}\), \(-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}\), \(\frac{2}{\sqrt{3}} < x < \infty\)
   
   Sign of \( f''(x) \):
   
   - \( f''(x) > 0 \)
   - \( f''(x) < 0 \)
   - \( f''(x) > 0 \)

   Conclusion: Concave upward, Concave downward, Concave upward

5. Points of inflection: \( \left( \frac{2}{\sqrt{3}}, \frac{20}{9} \right) \)

Question 6

Find the points of inflection and discuss the concavity of the graph of the function.

\( f(x) = 3(x - 4)^3 \)

1. \( f'(x) = 9(x - 4)^2 \)
2. \( = 3(x - 4)^2 \)
3. \( f''(x) = 4(x - 4)(3(x - 4)) \)
4. \( = 4(x - 4)(x - 1) \)
5. \( = 4(x - 4)(3x - 6) \)
6. \( = 12(x - 4)(x - 2) \)
7. \( f''(x) = 12(x - 4)(x - 2) \)
8. \( = 0 \) when \( x = 2, 4 \).
9. Test interval: \(-\infty < x < 2\), \(2 < x < 4\), \(4 < x < \infty\)
   
   Sign of \( f''(x) \):
   
   - \( f''(x) > 0 \)
   - \( f''(x) < 0 \)
   - \( f''(x) > 0 \)

   Conclusion: Concave upward, Concave downward, Concave upward

10. Points of inflection: \((2, -16), (4, 0)\)
**Question 7**

Analyze and sketch a graph of the function. Label any intercepts, relative extrema, points of inflection, and asymptotes. Use a graphing utility to verify your results.

\[ y = x^3 - 3x^2 + 3 \]

1. \[ y' = 3x^2 - 6x \]
2. \[ y'' = 6x - 6 \]
3. \[ y'' = 0 \] when \[ x = 0, x = 2 \].
4. \[ y'' = 6(x - 1) \]
5. \[ y'' = 0 \] when \[ x = 1 \].

<table>
<thead>
<tr>
<th>Condition</th>
<th>y'</th>
<th>y''</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ x &lt; 0 ]</td>
<td>+</td>
<td>-</td>
<td>Increasing, concave down</td>
</tr>
<tr>
<td>[ x = 0 ]</td>
<td>3</td>
<td>0</td>
<td>Relative maximum</td>
</tr>
<tr>
<td>[ 0 &lt; x &lt; 1 ]</td>
<td>-</td>
<td>-</td>
<td>Decreasing, concave down</td>
</tr>
<tr>
<td>[ x = 1 ]</td>
<td>1</td>
<td>0</td>
<td>Point of inflection</td>
</tr>
<tr>
<td>[ 1 &lt; x &lt; 2 ]</td>
<td>-</td>
<td>+</td>
<td>Decreasing, concave up</td>
</tr>
<tr>
<td>[ x = 2 ]</td>
<td>-1</td>
<td>0</td>
<td>Relative minimum</td>
</tr>
<tr>
<td>[ 2 &lt; x &lt; \infty ]</td>
<td>+</td>
<td>+</td>
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</tr>
</tbody>
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**Question 8**

Analyze and sketch a graph of the function. Label any intercepts, relative extrema, points of inflection, and asymptotes. Use a graphing utility to verify your results.

\[ f(x) = \frac{x^2 + 1}{x} \]

1. \[ f'(x) = \frac{1}{x} \]
2. \[ f''(x) = 1 - \frac{1}{x^2} \]
3. \[ f''(x) = 0 \] when \[ x = \pm 1 \].
4. \[ f''(x) = \frac{2}{x^3} \neq 0 \]
5. Relative maximum: \( (1, 2) \)
6. Relative minimum: \( (1, 2) \)
7. Vertical asymptote: \[ x = 0 \]
8. Slant asymptote: \[ y = x \]
9. 

![Graph of f(x) = x^2 + 1/x]
**Question 9**

Analyze and sketch a graph of the function. Label any intercepts, relative extrema, points of inflection, and asymptotes. Use a graphing utility to verify your results.

\[ y = \frac{x^2}{x^2 + 3} \]

1. \[ y' = \frac{6x}{(x^2 + 3)^2} \]

2. \[ = 0 \text{ when } x = 0. \]

3. \[ y'' = \frac{18(1 - x^2)}{(x^2 + 3)^3} \]

4. \[ = 0 \text{ when } x = \pm 1. \]

5. **Horizontal asymptote:** \( y = 1 \)

6. | \( x \) | \( y' \) | \( y'' \) | Conclusion |
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>( -\infty &lt; x &lt; -1 )</td>
<td>-</td>
<td>-</td>
<td>Decreasing, concave down</td>
</tr>
<tr>
<td>( x = -1 )</td>
<td>( \frac{1}{3} )</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
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<td>( x = 0 )</td>
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7. [Graph of the function showing key points and behaviors]