Chapter 4 Mathematical Expectation

Mean of a Random Variable

**Definition.** Let $X$ be a random variable with probability distribution $f(x)$. The mean or expected value of $X$ is,

$$
\mu = \mu_x = E(x) = \sum_{all\ x} xf(x), \text{ if } X \text{ is a discrete random variable}
$$

$$
\mu = \mu_x = E(x) = \int_{-\infty}^{\infty} xf(x) dx, \text{ if } X \text{ is a continuous random variable}
$$

**Example 1.** A coin is biased so that a head is three times as likely to occur as a tail. Find the expected number of tails when this coin is tossed twice.

$$
P(T) + P(H) = 1; \ P(T) + 3P(T) = 1 \Rightarrow P(T) = \frac{1}{4} \text{ and } P(H) = \frac{3}{4}.
$$

$x$: number of tails; $x: 0, 1, 2$ and the corresponding probabilities are

$$
f(0) = P(HH) = P(H)P(H) = \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}
$$

$$
f(1) = P(TH) + P(HT) = \frac{1}{4} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{1}{4} = \frac{6}{16}
$$

$$
f(2) = P(TT) = P(T)P(T) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}.
$$

Therefore the probability distribution of $X$ is given as

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>$\frac{9}{16}$</td>
<td>$\frac{6}{16}$</td>
<td>$\frac{1}{16}$</td>
</tr>
</tbody>
</table>

and $\mu = \mu_x = E(x) = \sum_{all\ x} xf(x) = 0 \cdot \frac{9}{16} + 1 \cdot \frac{6}{16} + 2 \cdot \frac{1}{16} = \frac{1}{2}$.

**Example 2.** If a dealer’s profit in units of $5000, on a new automobile can be looked upon as a random variable $X$ having the density function

$$
f(x) = \begin{cases} 
2(1-x), & 0 < x < 1 \\
0, & \text{otherwise}
\end{cases}
$$

Find the average profit per automobile.

$$
\mu = E(x) = \int_{0}^{1} xf(x) dx = \int_{0}^{1} 2x(1-x) dx = \frac{1}{3}. \text{ Thus, } \frac{1}{3} \cdot 5000 = $1666 \text{ is the profit per automobile.}
$$
Example 3. Let $X$ be a random variable with the following probability distribution

<table>
<thead>
<tr>
<th>$x$</th>
<th>3</th>
<th>6</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{3}$</td>
</tr>
</tbody>
</table>

Find $\mu_{g(x)}$, where $g(x) = (2x + 1)^2$.

$$\mu = \mu_{g(x)} = E[g(x)] = \sum_{all\ x} g(x) f(x) = \sum_{all\ x} (2x + 1)^2 f(x) = \frac{1}{2} \cdot \frac{1}{6} + 169 \cdot \frac{1}{2} + 361 \cdot \frac{1}{3} = 217.33$$

Theorem. Let $X$ be a random variable with probability distribution $f(x)$. The mean or expected value of the random variable $g(x)$ is

$$\mu_{g(x)} = E[g(x)] = \sum g(x) f(x(x))$$

if $X$ is discrete and

$$\mu_{g(x)} = E[g(X)] = \int g(x) f(x) dx$$

if $X$ is continuous.

Definition. Let $X$ and $Y$ be random variables with joint probability distribution $f(x, y)$. The mean or expected value of the random variable $g(X, Y)$ is

$$\mu_{g(x,y)} = E[g(X, Y)] = \sum_{x} \sum_{y} g(x, y) f(x, y)$$

if $X$ and $Y$ are discrete, and

$$\mu_{g(x,y)} = E[g(X, Y)] = \int \int g(x, y) f(x, y) dx dy$$

if $X$ and $Y$ are continuous.

Variance and Covariance

The most important measure of variability of a random variable $X$ is obtained by letting $g(x) = (x - \mu)^2$ in the mean formula. Because of its importance in statistics it is referred to as the variance of the random variable $X$, denoted by $\text{Var}(X)$ or $\sigma^2_X$ and defined as

$$\text{Var}(X) = \sigma^2_X = E[(X - \mu)^2] = \begin{cases} \sum (x - \mu)^2 f(x), & \text{if } X \text{ is discrete} \\ \int (x - \mu)^2 f(x) dx, & \text{if } X \text{ is continuous} \end{cases}$$

The positive square root of the variance is called the standard deviation $(\sigma_X)$ of $X$. 
**Theorem.** The variance of a random variable $X$ is

$$Var(X) = E(X^2) - [E(X)]^2 = E(X^2) - \mu_X^2.$$ 

**Definition.** Let $X$ and $Y$ be random variables with joint probability distribution $f(x, y)$. The covariance of $X$ and $Y$ is

$$Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

$$= \begin{cases} 
\sum\sum (x - \mu_X)(y - \mu_Y) f(x, y), & X \text{ and } Y \text{ are discrete} \\
\int\int (x - \mu_X)(y - \mu_Y) f(x, y) dx dy, & X \text{ and } Y \text{ are continuous} 
\end{cases}$$

**Theorem.** If $X$ and $Y$ are independent, then $Cov(X, Y) = 0$. The converse is not always true.

**Theorem.** The covariance of two random variables $X$ and $Y$ with means $\mu_X$ and $\mu_Y$ is

$$\sigma_{xy} = E(XY) - E(X)E(Y).$$

**Definition.** Let $X$ and $Y$ be random variables with covariance $\sigma_{xy}$ and standard deviations $\sigma_X$ and $\sigma_Y$, respectively. The correlation coefficient $X$ and $Y$ is

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_X \sigma_Y}.$$ 

**Example 4.** Suppose that $X$ and $Y$ are independent random variables having the joint probability distribution

<table>
<thead>
<tr>
<th>$f(x, y)$</th>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>$h(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.10</td>
<td>0.15</td>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.20</td>
<td>0.30</td>
<td>0.50</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.10</td>
<td>0.15</td>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>$g(x)$</td>
<td></td>
<td>0.40</td>
<td>0.60</td>
<td>1</td>
</tr>
</tbody>
</table>

Show that $\sigma_{xy}$ is zero.

By the above theorem $\sigma_{xy} = E(XY) - E(X)E(Y)$. To compute covariance, we need to find $E(XY)$, $E(X)$ and $E(Y)$.
\[ E(XY) = \sum_x \sum_y xyf(x, y) = 2.1(0.10) + 4.1(0.15) + 2.3(0.20) + 4.3(0.30) + 2.5(0.10) + 4.5(0.15) = 9.6 \]

\[ E(x) = \sum_x xg(x) = 2(0.40) + 4(0.60) = 3.2 \]

\[ E(Y) = \sum_y yh(y) = 1(0.25) + 3(0.50) + 5(0.25) = 3.0 \]

Therefore \( \sigma_{XY} = E(XY) - E(X)E(Y) = 9.6 - (3.2)(3.0) = 0 \).

**Example 5.** Find the covariance of the random variables \( X \) and \( Y \) having the joint probability density

\[ f(x, y) = \begin{cases} 
  x + y, & 0 < x < 1, 0 < y < 1 \\
  0, & \text{otherwise}
\end{cases} \]

**The Mean and Variance of Linear Combinations of Random Variables**

(1) \( E(aX + b) = aE(X) + b, a, b \in \mathbb{R} \)

(2) Let \( X \) and \( Y \) be two random variables, then

\[ E(X \pm Y) = E(X) \pm E(Y) \]

(3) If \( X \) and \( Y \) are independent, then

\[ E(XY) = E(X)E(Y) \]

(4) If \( a \) and \( b \) are constants, then

\[ \sigma_{aX+bY}^2 = \text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2ab \text{Cov}(X,Y) = a^2 \sigma_X^2 + b^2 \sigma_Y^2 + 2ab \sigma_{XY} \]

(5) If \( X \) and \( Y \) are independent, then

\[ \text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) \]

**Example 6.** If \( X \) and \( Y \) are random variables with variances \( \sigma_X^2 = 2 \) and \( \sigma_Y^2 = 4 \) and covariance \( \sigma_{XY} = -2 \), find the variance of the random variables \( Z = 3X - 4Y + 8 \).
\[ Var(Z) = Var(3X - 4Y + 8) \]
\[ = 9Var(X) + 16Var(Y) + 23(-4)Cov(X,Y) \]
\[ = 9\sigma_X^2 + 16\sigma_Y^2 - 24\sigma_{XY} \]
\[ = 9.2 + 16.4 - 24(-2) \]
\[ = 130 \]

**Example 7.** If \( X \) and \( Y \) are independent random variables with variances \( \sigma_X^2 = 5 \) and \( \sigma_Y^2 = 3 \), find the variance of the random variables \( Z = -2X + 4Y - 3 \).

\[ Var(Z) = Var(-2X + 4Y - 3) \]
\[ = 4Var(X) + 16Var(Y) \]
\[ = 4\sigma_X^2 + 16\sigma_Y^2 \]
\[ = 4.5 + 16.3 \]
\[ = 68 \]

**Exercises**

**Exercise 1.** Let \( X \) denote the number of times a certain numerical control machine will malfunction: 1, 2, or 3 time on a given day. Let \( Y \) denote the number of times a technician is called on an emergency call. Their joint probability distribution is given as

<table>
<thead>
<tr>
<th>( f(x,y) )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>2</td>
<td>0.05</td>
<td>0.10</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0.20</td>
</tr>
</tbody>
</table>

Determine the covariance between \( X \) and \( Y \).

**Exercise 2.** Let \( X \) denote the number of heads and \( Y \) the number of heads minus the number of tails when 3 coins are tossed.

(a) Find the joint probability distribution of \( X \) and \( Y \).
(b) Find the marginal distributions of \( X \) and \( Y \).
(c) Find \( E(X), E(Y) \) and \( E(XY) \)
(d) Determine whether \( X \) and \( Y \) independent or not.

**Exercise 3.** Find the correlation coefficient between \( X \) and \( Y \) having the joint density function

\[ f(x,y) = \begin{cases} 
  x + y, & 0 < x < 1, \ 0 < y < 1 \\
  0, & \text{elsewhere}
\end{cases} \]
Exercise 4. Suppose that $X$ and $Y$ are independent random variables with probability densities

$$
g(x) = \begin{cases} 
\frac{8}{x^3}, & x > 2 \\
0, & \text{elsewhere}
\end{cases}$$

and

$$
h(y) = \begin{cases} 
2y, & 0 < y < 1 \\
0, & \text{elsewhere}
\end{cases}$$

Find the expected value of $Z = XY$. 