EASTERN MEDITERRANEAN UNIVERSITY
DEPARTMENT OF MATHEMATICS
Math322 – Probability and Statistical Methods
2010 – 2011 Spring Semester
Second Midterm Exam.
Date: 02.05.2011; Duration: 90 min.; Note: Calculator is not allowed.

1. Solve the following questions:
   A) The passengers leaving an airport can pass through any one of the three gates; A, B and C. Assuming that a passenger is equally likely to select any one the three gates, find the probability that among 7 passengers; 2 select Gate A, 3 select Gate B and 2 select Gate C.

   \[ P(2A,3B,2C) = \frac{7!}{2!3!2!} \times \left(\frac{1}{3}\right)^2 \times \left(\frac{1}{3}\right)^3 \times \left(\frac{1}{3}\right)^2 = \frac{7 \times 5 \times 4}{2 \times 3 \times 2 \times 3 \times 3 \times 3 \times 3} = \frac{70}{729} \]

   B) The average number of telephone calls coming into the central switchboard of an office building is 4 per minute. Find the probability that at least two calls will arrive in a given minute.

   \[
   P(X \geq 2) = 1 - \left[ f(0) + f(1) \right] = 1 - \left[ e^{-4} + e^{-4} \right] = 1 - \left[ \frac{e^{-4}}{0!} + \frac{e^{-4}}{1!} \right] = 1 - 5e^{-4}
   \]

   C) A lot of 1000 industrial products contains 50 defectives. If 10 of these industrial products are randomly selected, determine the probability that none of them will be defective.

   \[
   \left( \begin{array}{c} 950 \cr 10 \end{array} \right) \left( \begin{array}{c} 50 \cr 0 \end{array} \right) \approx h(0,10;\frac{1}{20}) = \left( \begin{array}{c} 10 \cr 0 \end{array} \right) \left( \begin{array}{c} 19 \cr 0 \end{array} \right) \left( \begin{array}{c} 19 \cr 1 \end{array} \right) = \left( \begin{array}{c} 19 \cr 0 \end{array} \right) \]
2. Consider the following joint density function
\[ f(x, y) = \begin{cases} 6y, & 0 < x < 1 \text{ and } 0 < y < 1 - x \\ 0, & \text{otherwise} \end{cases} \]

a) Show that \[ \iint_{R} f(x, y) \, dy \, dx = 1. \] (10 p.)

\[ 6 \int_{0}^{1} \int_{0}^{1-x} y \, dy \, dx = 3 \int_{0}^{1} \frac{1}{2}y^2 \big|_{0}^{1-x} \, dx = 3 \int_{0}^{1} (1 - x)^2 \, dx = 3 \left[ \frac{1}{3}x^3 \right]_{0}^{1} = 1 \]

b) Find \( P(Y > X) \). (15 p.)

\[ P(Y > X) = 6 \int_{0}^{1} \int_{0}^{1-x} y \, dy \, dx = 3 \int_{0}^{1} \frac{1}{2}y^2 \big|_{0}^{1-x} \, dx = 3 \int_{0}^{1} (1 - x)^2 \, dx = 3 \left[ \frac{1}{3}x^3 \right]_{0}^{1} = 3 \left[ \frac{1}{2} \right] = \frac{3}{2} \]

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c) Find the marginal distribution of \( X \); \( f(x) \). (10 p.)
\[ f(x) = 60 \int_0^x \int_0^{1-x} y^2 \, dy \, dx = 3(1-x)^2, \quad 0 \leq x < 1 \]
\[ f(x) = \begin{cases} 
3(1-x), & 0 \leq x < 1 \\
0, & \text{elsewhere} 
\end{cases} \]

d) Find the expectation \( E(3X - 2). \) (10 p.)

\[
E(X) = \int_0^1 \int_0^{1-x} xy \, dy \, dx = \int_0^1 x(1-x) \, dx = \int_0^1 (x - 2x^2 + x^3) \, dx = \left[ \frac{1}{2}x^2 \right]_0^1 + \left[ \frac{2}{3}x^3 \right]_0^1 + \left[ \frac{1}{4}x^4 \right]_0^1 = \frac{1}{4}
\]

\[
E(3X - 2) = 3E(X) - 2 = 3 \left( \frac{1}{4} \right) - 2 = -\frac{5}{4}
\]
3. Suppose $X$ and $Y$ are two independent discrete random variables with following distributions; $f(x)$ and $g(y)$, respectively.

\[
\begin{array}{c|ccc}
 y & -1 & 0 & 1 \\
 g(y) & 0.3 & 0.5 & 0.2 \\
\end{array}
\quad
\begin{array}{c|cc}
 x & 1 & 2 \\
 f(x) & 0.7 & 0.3 \\
\end{array}
\]

a) Construct the joint distribution of $X$ and $Y$: $f(x, y)$, with marginal distributions. (10 p.)

\[
\begin{array}{c|ccc}
 x(y) & -1 & 0 & 1 \\
 f(x) & 0.21 & 0.35 & 0.14 \\
\end{array}
\quad
\begin{array}{c|ccc}
 y(x) & 0.9 & 0.15 & 0.06 \\
 g(y) & 0.3 & 0.5 & 0.2 \\
\end{array}
\]

\[
h(1,-1)=f(1)g(-1)=0.21 \\
h(1,0)=f(1)g(0)=0.35 \\
h(1,1)=f(1)g(1)=0.14 \\
h(2,-1)=f(2)g(-1)=0.09 \\
h(2,0)=f(2)g(0)=0.15 \\
h(2,1)=f(2)g(1)=0.06 \\
\]

b) Find $P(X + Y \leq 1) = 0$. (6 p.)

$P(X + Y \leq 1) = h(1, -1) + h(1, 0) + h(2, -1) = 0.21 + 0.35 + 0.09 = 0.65$

c) Show that $\text{Cov}(X, Y) = 0$. (10 p.)

\[
\text{cov}(X, Y) = E(XY) - E(X)E(Y) \\
E(XY) = 1*(-1)(0.21) + 1*1(0.14) + 2*(-1)(0.09) + 2*1(0.06) = -0.13 \\
E(X) = 1(0.7) + 2(0.3) = 1.3 \\
E(Y) = -1(0.3) + 1(0.2) = -0.1 \\
\text{cov}(X, Y) = E(XY) - E(X)E(Y) = -0.13 + 0.13 = 0
\]

d) Find the correlation coefficient of $X$ and $Y$: $\rho(X, Y)$, and identify it. (4 p.)

\[
\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{0}{\sigma_X \sigma_Y} = 0 \quad \text{No Correlations}
\]