(1) (a) Determine for what values of \( k \in \mathbb{R} \) the linear system \( \begin{aligned} x + ky &= 1 \quad \text{has no solution, unique solution and infinitely many solutions. When the system has solution(s), write down the (general) solution.} \\
kx + y &= 1 \end{aligned} \)

Solution: By performing row-operations in the augmented matrix of the given linear system we have successively

\[
\begin{bmatrix}
1 & k & : & 1 \\
k & 1 & : & 1 \\
\end{bmatrix} - kr_1 + r_2 \rightarrow r_2
\]

If \( k = 1 \) the initial linear system is equivalent with \( x + y = 1 \) and it has infinitely solutions \( x = 1 - t, \ y = t, \ t \in \mathbb{R} \), in this case. If \( k = -1 \) the last matrix in the above row-reduction process is

\[
\begin{bmatrix}
1 & -1 & : & 1 \\
0 & 0 & : & 2 \\
\end{bmatrix}
\]

and it shows that, in this case, the linear system has no solutions. Finally, if \( k \in \mathbb{R} \setminus \{-1, 1\} \), then the row-reduction process can be continued in the following way

\[
\begin{bmatrix}
1 & k & : & 1 \\
0 & 1 - k^2 & : & 1 - k \\
\end{bmatrix} \xrightarrow{1/1-k^2} \begin{bmatrix}
1 & k & : & 1 \\
0 & 1 & : & \frac{1}{1+k} \\
\end{bmatrix} \xrightarrow{kr_2 + r_1 - r_3} \begin{bmatrix}
1 & 0 & : & \frac{1}{1+k} \\
0 & 1 & : & \frac{1}{1+k} \\
\end{bmatrix}
\]

the last matrix in the above row-reduction process shows that the initial linear system has, in this case, the unique solution \( x = y = \frac{1}{1+k} \).

(b) For which values of \( x \in \mathbb{R} \), is the matrix \( A = \begin{bmatrix}
1 & 1 & x \\
1 & x & x \\
x & x & x \\
\end{bmatrix} \) invertible?

Solution: The given matrix \( A \) is invertible if and only if \( \det(A) \neq 0 \). But \( \det(A) =
\[
\begin{vmatrix}
1 & 1 & x \\
1 & x & x \\
x & x & x \\
\end{vmatrix} = 0 \iff x^2 + x^2 + x^2 - x^3 - x^2 - x = 0 \iff 2x^2 - x^3 - x = 0 \iff x(x^2 - 2x + 1) = 0 \iff x(x - 1)^2 = 0 \iff x \in \{0, 1\}. \]

Therefore \( A \) is invertible if and only if \( x \in \mathbb{R} \setminus \{0, 1\} \).

(c) Let \( A = \begin{bmatrix}
2 & -1 \\
3 & 1 \\
\end{bmatrix} \).

(i) Find elementary matrices \( E_1, E_2, E_3, E_4 \) such that \( E_1E_2E_3E_4A = I_2 \);

(ii) Write \( A \) as a product of elementary matrices.

Solution: By performing row-reduction operations in the given matrix we have successively:
Thus, the required elementary matrices can be obtained in the following way:

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The inverses of $E_4, E_3, E_2, E_1$ are respectively $E_4^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $E_3^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 5/2 \end{bmatrix}$, $E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and obviously $A = E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1}$.

(2) (a) Prove that the matrices $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$ commute if and only if

\[
\begin{vmatrix} b & a \\ e & d \end{vmatrix} = \begin{vmatrix} b & c \\ e & f \end{vmatrix}.
\]

Solution: We have successively: $AB = BA \iff \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} da + ae + bf \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} da & db + ec \\ 0 & fc \end{bmatrix} \iff ae + bf = db + ec \iff bd - ae = bf - ce \iff \begin{vmatrix} b & a \\ e & d \end{vmatrix} = \begin{vmatrix} b & c \\ e & f \end{vmatrix}.$

(b) If $u, v \in \mathbb{R}^n$ are such that $||u + v|| = 1$ and $||u - v|| = 5$, find $u \cdot v$.

Solution: $||u + v|| = 1 \Rightarrow ||u + v||^2 = 1 \iff u^2 + 2u \cdot v + v^2 = 1$ $||u - v|| = 5 \Rightarrow ||u - v||^2 = 25 \iff u^2 - 2u \cdot v + v^2 = 25$. By subtracting the obtained relations one can get that $4u \cdot v = -24$ or, equivalently $u \cdot v = -6$.

(c) Decide whether the following mappings are linear and justify the answer in each case

(i) $T_1 : \mathbb{R}^2 \to \mathbb{R}$, $T_1(x, y) = x + y$;

(ii) $T_2 : \mathbb{R}^2 \to \mathbb{R}$, $T_2(x, y) = |x| + |y|$.

Solution: $T_1$ is linear because $T_1((x, y) + (x', y')) = T_1(x + x', y + y') = (x + x) + (y + y') = (x + y) + (x' + y') = T_1(x, y) + T_1(x', y')$ for all $(x, y), (x', y') \in \mathbb{R}^2$ and also $T_1(k(x, y)) = T_1(kx, ky) = kx + ky = k(x + y) = kT_1(x, y)$ for all $k \in \mathbb{R}$ and all $(x, y) \in \mathbb{R}^2$. 

2
$T_2$ is not linear because $T_2(-1, -1) = |-1| + |-1| = 2 \neq -2 = T_2(1, 1)$.

$$x + y + z = 1$$

(d) Solve the following linear system $-x + 2y + 3z = 1$ by Cramer’s rule.

$$x + 4y + 9z = 1$$

Solution: The determinant of the coefficient matrix $A$ of the given linear system

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & 3 & 8 \end{vmatrix} = 24 - 12 = 12 \neq 0,$$

which means that the system can be indeed solved by the Cramer’s rule and

$$x = \frac{1}{12} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \frac{1}{12} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = \frac{1}{6};$$

$$y = \frac{1}{12} \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 3 \\ 1 & 1 & 9 \end{vmatrix} = \frac{1}{12} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = \frac{4}{3};$$

$$z = \frac{1}{12} \begin{vmatrix} 1 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 4 & 1 \end{vmatrix} = \frac{1}{12} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -\frac{1}{2}.$$

(3) (a) Decide whether the following subsets

(i) $A = \{(a, b, c, d) \in \mathbb{R}^4 \mid a - b = 2\},$

(ii) $B = \{(a, b, c, d) \in \mathbb{R}^4 \mid c = a + 2b$ and $d = a - 3b\},$

are subspaces of $\mathbb{R}^4$ and justify the answer in each case.

Solution: $A$ is not a subspace of $\mathbb{R}^4$ because $0 \not\in \mathbb{R}^4$, while $B$ is a subspace of $\mathbb{R}^4$ being the solution set of a linear system.

(4) Find a basis of the subspace $W = \text{Span}\{v_1, v_2, v_3, v_4, v_5\}$ of $\mathbb{R}^3$, where $v_1 = (1, 0, 1), v_2 = (0, 1, 1), v_3 = (1, 1, 2), v_4 = (1, 2, 1), v_5 = (-1, 1, -2)$.

Solution: We will find a solution of the given subspace of $\mathbb{R}^3$ by performing row-reduction operations in the following matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & -2 \end{bmatrix} \xrightarrow{-r_1 + r_3} \begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 & -1 \end{bmatrix} \xrightarrow{-r_2 + r_3} \begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{-2r_3 + r_2}$$
\[
\begin{bmatrix}
1 & 0 & 1 & 1 & -1 \\
0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 & 0 & -2 \\
0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix},
\]
which shows that \( \{v_1, v_2, v_4\} \) is a basis of \( W \).

(5) Given \( A = \begin{bmatrix}
1 & 1 & 4 & 1 & 2 \\
0 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 2 \\
1 & -1 & 0 & 0 & 2 \\
2 & 1 & 6 & 0 & 1
\end{bmatrix} \), find:

(a) a basis and the dimension of the row space of \( A \);
(b) a basis and the dimension of the column space of \( A \);
(c) a basis and the dimension of the null space of \( A \).

Solution: By performing row-reduction operations in the given matrix we have successively:

\[
\begin{bmatrix}
1 & 1 & 4 & 1 & 2 \\
0 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 2 \\
1 & -1 & 0 & 0 & 2 \\
2 & 1 & 6 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 4 & 1 & 2 \\
0 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 2 \\
0 & -2 & -4 & -1 & 0 \\
0 & -1 & -2 & -2 & -3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 4 & 1 & 2 \\
0 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 4 & 0 & 0 \\
0 & 1 & 2 & 0 & -1 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 2 & 0 & 1 \\
0 & 1 & 2 & 0 & -1 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]
Thus a basis for the row space is \( \{(1, 0, 2, 0, 1), (0, 1, 2, 0, -1), (0, 0, 1, 2)\} \) so that its dimension is 3. Also a basis for the column space is

\[
\begin{bmatrix}
1 & 1 \\
0 & 1 \\
1 & -1 \\
2 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 1 \\
1 & -1 \\
0 & 0
\end{bmatrix}
\]

its dimension being consequently 3.

The null space of the given matrix is the solution set of the corresponding homogeneous linear system of the given matrix, which is in its turn equivalent with the linear system that corresponds to the last matrix in the above row-reduction process. Thus the required null space is

\[
x_1 = -2t - s \\
x_2 = -2t + s \\
x_3 = t \\
x_4 = -2s \\
x_5 = s
\]

being the solution set of the corresponding linear system of the given matrix, which is in its turn equivalent with the linear system that corresponds to the last matrix in the above row-reduction process. Thus the required null space is

\[
\begin{bmatrix}
-2t - s \\
-2t + s \\
t \\
-2s \\
s
\end{bmatrix}, t, s \in \mathbb{R}
\]

which means that the required null space is

\[
\text{Span}\left\{ \begin{bmatrix}
-2 \\
-2 \\
1 \\
0 \\
0
\end{bmatrix}, t, s \in \mathbb{R}
\right\}
\]

Because the vectors

\[
\begin{bmatrix}
1 \\
0 \\
-2 \\
1
\end{bmatrix}, t, s \in \mathbb{R}
\]

are obviously linearly independent, it follows that they form a basis of the required null space, its dimension being consequently 2.

(1) Show that the mapping \( M_{22} \times M_{22} \rightarrow \mathbb{R} \), \( (U, V) \mapsto \langle U, V \rangle \) where

\[
\langle U, V \rangle = u_1v_1 + u_2v_3 + u_3v_2 + u_4v_4,
\]

whenever \( U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \), \( V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} \), is not an inner product on \( M_{22} \).

**Solution:** The given mapping is indeed not an inner product since

\[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} = 0 \cdot 0 + 1 \cdot (-1) + (-1) \cdot 1 + 0 \cdot 0 = -2 < 0.
\]
(2) Find the values of \(a, b, c \in \mathbb{R}\) for which the matrix \(A\) is symmetric, where

\[
A = \begin{bmatrix}
2 & a + b + c & 4b - 3c \\
5 & 1 & 2a + c \\
2 & 4 & 6
\end{bmatrix}.
\]

\(a + b + c = 5\)

**Solution:** The required values are solutions of the linear system \(4b - 3c = 2\)

\(2a + c = 4\)

that can be solved by performing row-reduction operations in the corresponding augmented matrix, namely:

\[
\begin{bmatrix}
1 & 1 & 1 & : & 5 \\
0 & 4 & -3 & : & 2 \\
2 & 0 & 1 & : & 4
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 & 1 & : & 5 \\
0 & 4 & -3 & : & 2 \\
0 & -2 & -1 & : & 6
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 & 1 & : & 5 \\
0 & 1 & -3/4 & : & 1/2 \\
0 & -2 & -1 & : & 6
\end{bmatrix}
\]

The corresponding linear system of the last matrix in the above row-reduction process, which is equivalent with the initial one, has the unique solution \(a = 1, b = c = 2\).

(3) Let \(u_1 = (0, -1, -1), u_2 = (-1, 0, 1), u_3 = (1, 1, 1) \in \mathbb{R}^3\).

(a) Show that \(\{u_1, u_2, u_3\}\) is a basis of \(\mathbb{R}^3\);

(b) Apply the Gram-Schmidt process to convert the basis \(\{u_1, u_2, u_3\}\) into an orthogonal basis of \(\mathbb{R}^3\) and find its corresponding orthonormal basis.

**Solution:** \(v_1 = u_1 = (0, -1, -1)\):

\(v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1 = (0, -1, 1) - \frac{1}{2} (0, -1, -1) = (-1, -\frac{1}{2}, \frac{1}{2})\)

\(v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2 = (1, 1, 1) - \frac{2}{2} (0, -1, -1) - \frac{2}{3} (1, -\frac{1}{2}, \frac{1}{2}) = (1, 0, 0) + (2/3, -3/4, 3/4) = (\frac{5}{3}, -\frac{1}{4}, \frac{3}{4}).\)

Thus \(v_1 = (0, -1, -1), v_2 = (-1, -\frac{1}{2}, \frac{1}{2}), v_3 = (\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})\) form an orthonormal basis of \(\mathbb{R}^3\). The norms of these vectors are:

\(\|v_1\| = \sqrt{2}, \|v_2\| = \sqrt{\frac{3}{2}}, \|v_3\| = \frac{1}{\sqrt{3}}\)

such that the corresponding orthonormal basis is

\(q_1 = \frac{v_1}{\|v_1\|} = (0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), q_2 = \frac{v_2}{\|v_2\|} = (-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}), q_3 = \frac{v_3}{\|v_3\|} = (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}).\)