1) For $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 4, 8\}$, $C = \{1, 2, 3, 5, 7\}$, and $D = \{2, 4, 6, 8\}$. Determine each of the following:

(a) $(A \cup B) \cap C = \{1, 2, 3, 4, 5\}$
(b) $A \cup (B \cap C) = \{1, 2, 3, 4, 5\}$
(c) $C \cup D = \{1, 3, 4, 5, 6, 7, 8, 9, 10\}$
(d) $(A \cup B) - C = \{4, 8\}$
(e) $A \cup (B - C) = \{1, 2, 3, 4, 5\}$
(f) $(B - C) - D = \{\}$
(g) $B - (C - D) = \{2, 4, 8\}$
(h) $(A \cup B) - (C \cap D) = \{1, 3, 4, 5, 8\}$
(i) $A \oplus B = \{3, 5, 8\}$

2) Given that

$U = \{x \mid x$ is a positive integer less than 20\}$
$A = \{1, 5, 9, 19\}$
$B = \{b \mid b$ is a positive odd integer less than 11\}$
$C = \{c \mid c$ is a positive odd integer less than 20\}$

(a) $A \cup B = \{1, 3, 5, 7, 9, 19\}$
(b) $A \cap B \cap C = \{1, 5, 9\}$
(c) $(A \cup C)' = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$
(d) $(A \cup B) \cap C = \{3, 4, 5, 7, 9, 10\}$

3) The universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8, 10\}$, $C = \{3, 4, 5, 7, 9, 11\}$. Find the followings.

(a) $A \cup B = \{\}$
(b) $B - (A \cup C) = \{8, 10\}$
(c) $(B - A) \cup C = \{3, 4, 5, 7, 9, 10, 11\}$
(d) $A \cap C = \{2, 4, 6\}$
(e) $A' = \{7, 8, 9, 10, 11\}$
(f) Draw the VENN DIAGRAM of these sets and find $(A \cup B) - C$ and $B'$.

4) Given the Universal set $U = \{\text{positive integers not larger than 12}\}$, and the sets

$A = \{\text{positive integers not more than 6}\}$
$B = \{3, 4, 6, 7\}$
$C = \{5, 6, 7, 8, 9, 10\}$

Find

(i) $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$
(ii) $(A \cup B) / C = \{1, 2, 3, 4\}$
(iii) $|A - B| = 3$
(iv) $P(A-B) = \{\phi, 1, 2, 5, \{1, 2\}, \{1, 5\}, \{2, 5\}, \{1, 2, 5\}\}$

5) Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5\}$.

(a) List all subsets of $A$ that are disjoint from $B$.

$\{\phi, 2, 4, \{2, 4\}\}$

(b) List at least three subsets of $A$ that are incomparable with $B$.

use the definition of incomparable sets…

(c) How many proper subsets of $B$ are there?

$n(B) = 2, 2^n - 1 = 7$

d) Find $n(A \cup B) = 5$

6) In a survey of 100 students, the following data was collected. There were 19 taking biology, 20 taking chemistry, 19 taking physics, 7 taking physics and chemistry, 8 taking biology and chemistry, 9 taking biology and physics, and 5 taking all three subjects.

(a) How many of the group were not taking any of three subjects? 61
(b) How many were taking only chemistry? 10
(c) How many were taking physics and chemistry, but not biology? 20
7) How many bit strings with length 8 either start with 1 or end with 00?
\[ |A1| = \text{number of length 8 bitstrings that start with 1} = 2^7 \]
\[ |A2| = \text{number of length 8 bitstrings that end with 00} = 2^6 \]
\[ |A2 \cup A1| = \text{number of length 8 bitstrings that end with 00 and start with 1} = 2^5 \]
Thus: \[ |A1 \cup A2| = 2^7 + 2^6 - 2^5 \]

9) Let \( X = \{1,2,3,4,5,6,7,8,9,10,12,14,16\} \) and let \( R \) be a relation on \( X \) defined as
\( R = \{(a,b) : a-b \text{ is a multiple of 5}\} \). \( R = \{(1,6),(6,1),(2,7),(7,2),(3,8),(8,3),(4,9),(9,4),(5,10),(10,5),(7,12),(12,7),(9,14),(14,9)\} \)
(a) Show that \( R \) is an equivalence relation.
(i) \((a,a) \in R, \forall a \in X\) since \( a-a=0 \) is a multiple of 5, \( R \) is reflexive
(ii) if \((a,b) \in R\) then \((b,a) \in R\), \( a-b = 5k \) then \( b-a = -(a-b) = -5k \) is a multiple of 5, \( R \) is symmetric
(iii) if \((a,b) \in R, and (b,c) \in R\), then \((a,c) \in R\), \( a-b = 5k_1 \) and \( b-c = 5k_2 \), then \( (a-b) + (b-c) = (a-c) = 5k_1 + 5k_2 = 5(k_1 + k_2) \) is a multiple of 5, \( R \) is transitive
\( R \) is an equivalence relation

(b) Write the distinct equivalence classes.
\( C(0), C(1), C(2), C(3), C(4) \)

10) Let \( Z \) be the set of integers. For any two numbers \( a, b \) in \( Z \), if \( a-b \) is even (divisible by 2).
(a) Show that this is an equivalence relation
(i) \( a \equiv a \) for all \( a \in Z \) since \( a-a=0 \) is an even number.
(ii) If \( a \equiv b \), then \( a-b \) is even, then \( b-a = -(a-b) \) is also even, so \( b \equiv a \).
(iii) If \( a \equiv b \) and \( b \equiv c \), then \( a-b \) and \( b-c \) are even. Their sum is also an even number. So \( a-c = (a-b) + (b-c) \) is even and \( a \equiv c \).
We see that \( \equiv \) is an equivalence relation on \( Z \).

(b) How many equivalence classes does this equivalence relation have? List them all.
\( C(0) = \{-4,-2,2,4,..\} \)
\( C(1) = \{-3,-1,1,3,..\} \)

11) Let \( Z \) be the set of integers. For any two numbers \( a, b \) in \( Z \), if \( a-b \) is divisible by \( n \).
(a) Show that this is an equivalence relation
(i) \( a \equiv a \) for all \( a \in Z \) since \( a-a=0 = n0 \) is divisible by \( n \).
(ii) If \( a \equiv b \), then \( a-b = nm \) for some \( m \in Z \), so \( b-a = -(a-b) = n(-m) \) is divisible by \( n \), and \( b \equiv a \).
(iii) If \( a \equiv b \) and \( b \equiv c \), then \( a-b = nm \) and \( b-c = nk \) for some \( m,k \in Z \), so \( a-c = (a-b) + (b-c) = nm + nk = n(m+k) \) is divisible by \( n \), and so \( a \equiv b \).

(b) How many equivalence classes does this equivalence relation have?
There are \( n \) equivalence classes

12) Define \( \sim \) on \( Z \) by \( a \sim b \) iff \( 3a+b \) is multiple of 4
(a) Prove that \( \sim \) defines an equivalence relation
b) Find the equivalence class of 0
(b) \( \tilde{0} = \{ x \in Z \mid x \sim 0 \} = \{ x \mid 3x = 4k \text{ for some integer } k \} \). Now if \( 3x = 4k \), \( k \) must be a multiple of 3. So \( 3x = 12\ell \) for some \( \ell \in Z \) and \( x = 4\ell \). \( \tilde{0} = 4Z \).

13) For any \( a \) and \( b \), define \( a \sim b \) if \( 3a + 4b = 7n \) for some integer \( n \)
a) Prove that \( \sim \) defines an equivalence relation
*(a) Reflexive: For any \( a \in Z \), \( a \sim a \) because \( 3a + 4a = 7a \) and \( a \) is an integer.

Symmetric: If \( a, b \in Z \) and \( a \sim b \), then \( 3a + 4b = 7n \) for some integer \( n \). Then \( 3b + 4a = (7a + 7b) - (3a + 4b) = 7a + 7b - 7n = 7(a + b - n) \) and \( a + b - n \) is an integer. Thus \( b \sim a \).

Transitive: Suppose \( a, b, c \in Z \) with \( a \sim b \) and \( b \sim c \). Then \( 3a + 4b = 7n \) and \( 3b + 4c = 7m \) for some integers \( n \) and \( m \). Then \( 7n + 7m = (3a + 4b) + (3b + 4c) = (3a + 4c) + 7b \), so \( 3a + 4c = 7n + 7m - 7b = 7(n + m - b) \) and \( n + m - b \) is an integer. Thus \( a \sim c \).

b) Find the equivalence class of 0
(b) \( \tilde{0} = \{ x \in Z \mid x \sim 0 \} = \{ x \in Z \mid 3x = 7n \text{ for some integer } n \} \). Now if \( 3x = 7n \), \( n \) must be a multiple of 3. So \( 3x = 21k \) for some \( k \in Z \) and \( x = 7k \). We conclude that \( \tilde{0} = 7Z \).

14) Given \( A = \{1,2,3,4\} \). Consider the relation on \( A \), \( R = \{(1,1),(2,2),(3,3),(4,4),(1,2),(1,3),(2,3),(3,2),(2,1),(3,1)\} \)
a) Determine whether the \( R \) is reflexive? Explain.
\( R \) is reflexive, because \( (1,1),(2,2),(3,3),(4,4) \in R \).

b) Determine whether the \( R \) is symmetric? Explain.
\( R \) is symmetric, because \( (1,2),(2,1) \in R \), \( (2,3),(3,2) \in R \)

c) Determine whether the \( R \) is transitive? Explain.
\( R \) is transitive, because \( (1,2),(2,1) \in R \), \( (1,1) \in R \), \( (1,3),(3,2) \in R \), \( (1,2) \in R \)

d) Determine whether the \( R \) is antisymmetric? Explain.
\( R \) is not antisymmetric, because \( (1,2) \in R \), \( (2,1) \in R \)

15) Let \( X = \{1,2,3\} \) and \( Y = \{a,b,c,d\} \). Define \( H : X \to Y \) as follows: \( H(1) = c \), \( H(2) = a \) and \( H(3) = d \).
16) If \( A = \{a, b, c\} \), \( B = \{x, y\} \) and \( C = \{u, v, w\} \) and if \( f : A \rightarrow B \) and \( g : B \rightarrow C \) are the functions \( f = \{(a, x), (b, y), (c, x)\} \) and \( g = \{(x, u), (y, w)\} \). Find the composition \( g \circ f \).

\[
go f = \{(a, u), (b, w), (c, u)\}
\]

17) Let \( A = \{1, 2, 3, 4\} \), \( B = \{a, b, c, d, e\} \) and \( C = \{7, 8, 9\} \). Consider \( f : A \rightarrow B \) and \( g : B \rightarrow C \) as defined by \( f = \{(1, a), (2, b), (3, d), (4, b)\} \) and \( g = \{(a, 8), (b, 9), (c, 8), (d, 7), (e, 9)\} \). Find \( g \circ f \) and \( f \circ g \), if defined, if not defined explain why.

\( f \circ g \) is not defined, because the domain of \( f \) does not contain the range of \( g \).

18) If \( f \) and \( g \) are the functions \( f : R \rightarrow R \) defined by \( f(x) = 2x - 3 \) and \( g(x) = x^2 + 1 \). Find \( g \circ f \) and \( f \circ g \).

\[
g \circ f = (2x-3)^2 + 1 = 4x^2-12x+9+1 = 4x^2-12x+10
\]

\[
f \circ g = 2(x^2 + 1) - 3 = 2x^2 + 2 - 3 = 2x^2 - 1
\]

19) Given \( f(x) = 2x - 3 \) and \( g(x) = \frac{1}{2}(x + 3) \)

a) Find \( f \circ g (3), g \circ f (2) \)

\[
f(g(3)) = f(3) = 3
\]

\[
g(f(3)) = g(3) = 3
\]

b) Are they inverses of each other? if they are find \( I(5) \).

\[
fog(x) = 2\left(\frac{1}{2}(x+3)\right) - 3 = x + 3 - 3 = x
\]

\[
gof(x) = \frac{1}{2}(2x - 3 + 3) = x
\]

yes they are inverses

\( I(5) = fog(5) = 5 \)

20) Let \( A = \{x \in R \mid x \neq 2\} \) and \( B = \{x \in R \mid x \neq 1\} \). Define \( f : A \rightarrow B \) and \( f : B \rightarrow A \) by

\[
f(x) = \frac{x}{x - 2} \quad \quad g(x) = \frac{2x}{x - 1}
\]
Q) Show that the function \( f : R \to (1, \infty) \) and \( g: (1, \infty) \to R \) defined by
\[
f(x) = 3^{2x} + 1, \quad g(x) = \frac{1}{2} \log_3(x-1)
\]
are inverses.

Discrete mathematics 3e, page 84, problem 23

21) Let \( S = \{1,2,3,4,5\} \) and \( f, g, h: S \to S \) be functions defined by
\[
f = \{(1,2),(2,1),(3,4),(4,5),(5,3)\}
\]
\[
g = \{(1,3),(2,5),(3,1),(4,2),(5,4)\}
\]
\[
h = \{(1,2),(2,2),(3,4),(4,3),(5,1)\}
\]

(\textbf{a}) Find \( f \circ g \) and \( g \circ f \). Are these functions equal?

(\textbf{a}) \( [\text{BB}] \) \( f \circ g = \{(1,4),(2,3),(3,2),(4,1),(5,5)\} \);
\( g \circ f = \{(1,5),(2,3),(3,2),(4,4),(5,1)\} \)

Clearly, \( f \circ g \neq g \circ f \).

(\textbf{b}) Explain why \( f \) and \( g \) have inverses but \( h \) does not. Find \( f^{-1} \) and \( g^{-1} \).

(\textbf{b}) \( f^{-1} = \{(1,2),(2,1),(3,5),(4,3),(5,4)\} \);
\( g^{-1} = \{(1,3),(2,4),(3,1),(4,5),(5,2)\} \)

Functions \( f \) and \( g \) have inverses because they are one-to-one and onto while \( h \) does not have an inverse because it is not one-to-one (equally because it is not onto).

22) Let \( S = \{1,2,3,4,5\} \) and \( T = \{1,2,3,8,9\} \), and define functions \( f : S \to T \) and \( g : S \to S \)
\[
f = \{(1,8),(3,9),(4,3),(2,1),(5,2)\}
\]
\[
g = \{(1,2),(3,1),(2,2),(4,3),(5,2)\}
\]

(\textbf{a}) \( [\text{BB}] \) \( f \circ g = \{(1,1),(3,8),(2,1),(4,9),(5,1)\} \);
\( g \circ f \) is not defined because \( \text{rng } f = \{1,2,3,8,9\} \not\subseteq \text{dom } g = \{1,2,3,4,5\} \).
\( f \circ g \) is not defined because \( \text{rng } f = \{1,2,3,8,9\} \not\subseteq \text{dom } f = \{1,2,3,4,5\} \).
\( g \circ g = \{(1,2),(2,2),(3,2),(4,1),(5,2)\} \).

(\textbf{b}) \( f \) is one-to-one and onto;
\( g \) is not one-to-one: It contains both \( (1,2) \) and \( (2,2) \);
\( g \) is not onto: \( 4 \) is not in the range of \( g \).

(\textbf{c}) Since \( f \) is one-to-one and onto, it has an inverse: \( f^{-1} = \{(8,1),(9,3),(3,4),(1,2),(2,5)\} \).

(\textbf{d}) Since \( g \) is not one-to-one (or since \( g \) is not onto), \( g \) does not have an inverse.

(\textbf{e}) Find \( I(3) \) for function \( f \).
\( f \circ f \) is not defined so \( I(3) \) not defined.

23) Use mathematical induction to prove the truth of each of the following assertions for all \( n \geq 1 \)
\[8^n - 3^n \text{ is divisible by 5}\]
If $n = 1$, $8^n - 3^n = 8 - 3 = 5$ and $5 \mid 5$. Suppose the result is true for $n = k \geq 1$; that is, $5 \mid (8^k - 3^k)$ so that $8^k - 3^k = 5N$ for some integer $N$. Then

$$\begin{align*}
8^{k+1} - 3^{k+1} &= 8(8^k) - 3^{k+1} \\
&= 8(5N + 3^k) - 3^{k+1} \\
&= 40N + 8(3^k) - 3^{k+1} = 40N + 3^k(8 - 3) = 40N + 5(3^k) = 5(8N + 3^k)
\end{align*}$$

so that $5 \mid (8^{k+1} - 3^{k+1})$. By the Principle of Mathematical Induction, the result holds for all $n \geq 1$.

$5^{2n} - 2^{5n}$ is divisible by 7

If $n = 1$, $5^{2n} - 2^{5n} = 5^2 - 2^5 = 25 - 32 = -7$ is divisible by 7. Suppose $k \geq 1$ and the result is true for $k$, that is, assume $5^{2k} - 2^{5k}$ is divisible by 7. Thus $5^{2k} - 2^{5k} = 7N$ for some integer $N$. Then

$$\begin{align*}
5^{2(k+1)} - 2^{5(k+1)} &= 25(5^k) - 2^{5k+5} \\
&= 25(2^{5k} + 7N) - 32(2^{5k}) = -7(2^{5k}) + 25(7N) = 7(25N - 2^{5k})
\end{align*}$$

showing that $5^{2(k+1)} - 2^{5(k+1)}$ is divisible by 7. By the Principle of Mathematical Induction, the result holds for all $n \geq 1$.

$10^{n+1} + 10^n + 1$ is divisible by 3

**Basis step** $n = 1$ \hfill $10^1 + 10^0 + 1 = 111 \equiv 0 \pmod{3}$

**Inductive step** Assume $P(n)$ is true

Show that $P(n+1)$ is true.

\[
\begin{align*}
10^{n+2} + 10^{n+1} + 1 &= 10\cdot10^{n+1} + 10^1 + 1 \\
&= (1+9)10^{n+1} + (1+9) \cdot 10^1 + 1 \\
&= 10^{n+1} + 10^n + 1 + 9\left(10^{n+1} + 10^n\right) \\
&\equiv P(n) + 9P(n) \equiv 0 \pmod{3}
\end{align*}
\]

\[
\begin{align*}
1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 &= \frac{n(2n-1)(2n+1)}{3}
\end{align*}
\]
Prove that \(2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2\) for all \(n \in \mathbb{N}\).

I. We have \(2 = 2^1 + 1 - 2\), which proves the assertion for \(n = 1\).

II. Assume \(2 + 2^2 + 2^3 + \ldots + 2^k = 2^{k+1} - 2\). Now we must prove
\n\[2 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 2\]. We have
\n\[2 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} = (2^{k+1} - 2) + 2^{k+1}\] (by inductive hyp.)
\n\[= 2(2^{k+1}) - 2\]
\[= 2^{k+2} - 2\],
\nso the assertion is true for \(n = k + 1\) if it is true for \(n = k\). Thus
\[2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2\] for all \(n \in \mathbb{N}\).

Q) Given \(1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}\), \(n \geq 1\)

Find the sum of all the squares of the integers from 1 to 100(inclusive) that are non-multiples of 5.
= 100 \times 101 \times 201 - \left( 5^2 + 10^2 + 15^2 + \ldots + 100^2 \right)
= 2030100 - 5^2 (1^2 + 2^2 + 3^2 + \ldots + 20^2)
= 2030100 - 5^2 (20 \times 21 \times 41)
= 2030100 - 430500
= 1599600

24) The worst hockey team is allowed to draft one player from one of the top 3 teams. The top 3 teams have 20, 24, 22 players respectively.
a) How many different possible players can the worst team pick?
Solution: 20 + 24 + 22 = 66
b) Suppose the worst team can pick one player from each of the top 3 teams? How many different ways are there to pick the 3 players?
Solution: 20 \cdot 24 \cdot 22

25) How many license plates are available if each plate contains a sequence of 3 letters followed by 3 digits?
26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000

26) (a) How many bit strings of length 8 are possible?
(b) How many of these start with a 1 or end with 00?
(c) How many have at most two 0s?
Answers: (a) 256 (b) 160 (c) 37

27) How many functions are there from a set with 6 elements to a set with 4 elements.
Answer: 4096

28) How many one-to-one functions are there from a set with 6 elements to a set with
(a) 4 elements (b) 6 elements (c) 10 elements. Answers: (a) 0 (b) 720 (c) 151,200

28. Each user on a computer system has a password, which is six to eight characters long, where each character is a letter (case sensitive) or a digit. If each password must contain at least one digit and at least one letter, how many possible passwords are there?
Answer: 167,410,838,583,040 \leq 67 \quad 10^{14}