MATH152 CALCULUS II TUTORIAL – VIII

(27.04.2018)

Question 1:

Evaluate the iterated integral.

$$\int_1^2 \int_0^4 (x^2 - 2y^2 + 1)\, dx\, dy$$

1. $\int_1^2 \int_0^4 (x^2 - 2y^2 + 1)\, dx\, dy = \int_1^2 \left[\frac{1}{3}x^3 - 2xy^2 + x\right]_0^4 dy$
2. $= \int_1^2 \left(\frac{64}{3} - 8y^2 + 4\right) dy$
3. $= \frac{4}{3}\int_1^2 (19 - 6y^2)\, dy$
4. $= \left[\frac{4}{3}(19y - 2y^3)\right]_1^2$
5. $= \frac{20}{3}$

Question 2 :

Evaluate the iterated integral.

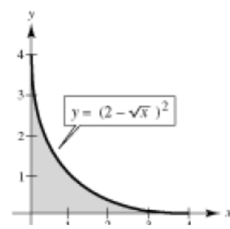
$$\int_0^1 \int_0^{\sqrt{1-y^2}} (x + y)\, dx\, dy$$

1. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x + y)\, dx\, dy = \int_0^1 \left[\frac{1}{2}x^2 + xy\right]_0^{\sqrt{1-y^2}} dy$
2. $= \int_0^1 \left[\frac{1}{2}(1 - y^2) + y\sqrt{1 - y^2}\right] dy$
3. $= \left[\frac{1}{2}y - \frac{1}{6}y^3 - \frac{1}{2}\left(\frac{2}{3}\right)(1 - y^2)^{3/2}\right]_0^1$
4. $= \frac{2}{3}$

Question 3 :

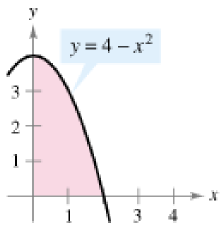
Use an iterated integral to find the area of the region bounded by the graphs of the equations.

$$\sqrt{x} + \sqrt{y} = 2, \quad x = 0, \quad y = 0$$

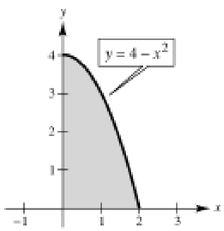
1. $\int_0^4 \int_0^{(2-\sqrt{x})^2} dy\, dx = \int_0^4 \left[y\right]_0^{(2-\sqrt{x})^2} dx$
2. $= \int_0^4 (4 - 4\sqrt{x} + x)\, dx$
3. $= \left[4x - \frac{8}{3}x\sqrt{x} + \frac{x^2}{2}\right]_0^4$
4. $= \frac{8}{3}$
5. $\int_0^4 \int_0^{(2-\sqrt{y})^2} dx\, dy = \frac{8}{3}$
6. Integration steps are similar to those above.
7. 

Question 4:

Use an iterated integral to find the area of the region.



1.



2. $A = \int_0^2 \int_0^{4-x^2} dy\, dx$

3. $= \int_0^2 \Big[y \Big]_0^{4-x^2} dx$

4. $= \int_0^2 (4 - x^2)\, dx$

5. $= \left[4x - \frac{x^3}{3} \right]_0^2$

6. $= \frac{16}{3}$

7. $A = \int_0^4 \int_0^{\sqrt{4-y}} dx\, dy$

8. $= \int_0^4 \Big[x \Big]_0^{\sqrt{4-y}} dy$

9. $= \int_0^4 \sqrt{4 - y}\, dy$

10. $= -\int_0^4 (4 - y)^{1/2}(-1)\, dy$

11. $= \left[-\frac{2}{3}(4 - y)^{3/2} \right]_0^4$

12. $= \frac{2}{3}(8)$

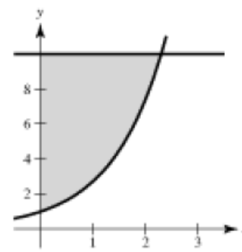
13. $= \frac{16}{3}$

Question 5:

Sketch the region R of integration and switch the order of integration.

$$\int_1^{10} \int_0^{\ln y} f(x, y)\, dx\, dy$$

1.



2. $\int_1^{10} \int_0^{\ln y} f(x, y)\, dx\, dy,\ 0 \le x \le \ln y, 1 \le y \le 10$

3. $= \int_0^{\ln 10} \int_{e^x}^{10} f(x, y)\, dy\, dx$

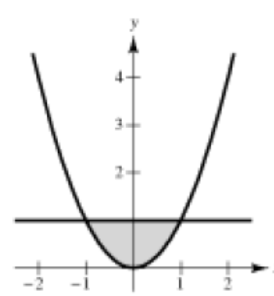
Question 6:

Sketch the region R of integration and switch the o

of integration.

$$\int_{-1}^{1} \int_{x^2}^{1} f(x, y)\, dy\, dx$$

1.



2. $\int_{-1}^{1} \int_{x^2}^{1} f(x, y)\, dy\, dx,\ x^2 \le y \le 1, -1 \le x \le 1$

3. $= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\, dx\, dy$

Question : 7

Use an iterated integral to find the area of the region bounded by the graphs of the equations.

$2x - 3y = 0, \quad x + y = 5, \quad y = 0$

1. $A = \int_0^3 \int_0^{2x/3} dy\,dx + \int_3^5 \int_0^{5-x} dy\,dx$
2. $= \int_0^3 \Big[y \Big]_0^{2x/3} dx + \int_3^5 \Big[y \Big]_0^{5-x} dx$
3. $= \int_0^3 \frac{2x}{3}\,dx + \int_3^5 (5 - x)\,dx$
4. $= \left[\frac{1}{3}x^2 \right]_0^3 + \left[5x - \frac{1}{2}x^2 \right]_3^5$
5. $= 5$
6. $A = \int_0^2 \int_{3y/2}^{5-y} dx\,dy$
7. $= \int_0^2 \Big[x \Big]_{3y/2}^{5-y} dy$
8. $= \int_0^2 \left(5 - y - \frac{3y}{2} \right) dy$
9. $= \int_0^2 \left(5 - \frac{5y}{2} \right) dy$

Question : 8

Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$..

Solution

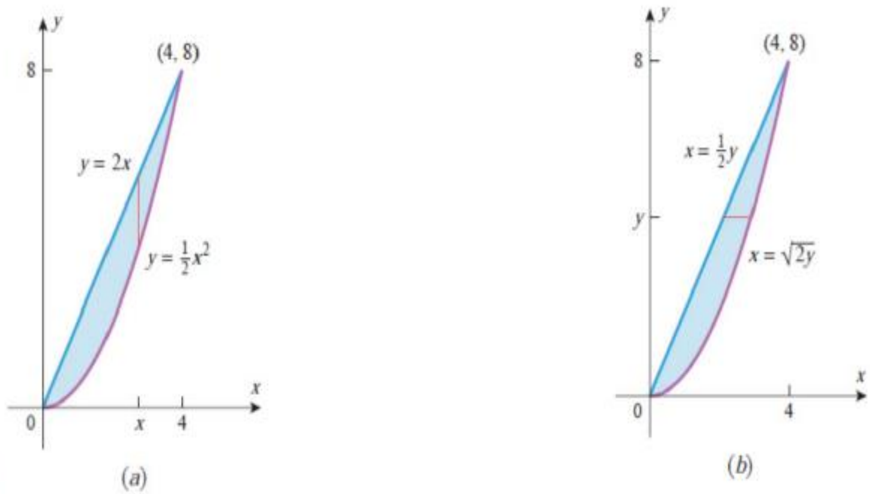
The region R may be treated equally well as type I (Figure 14.2.14*a*) or type II (Figure 14.2.14*b*).

Treating R as type I yields

$$\text{area of } R = \iint_R dA = \int_0^4 \int_{x^2/2}^{2x} dy\,dx = \int_0^4 [y]_{y=x^2/2}^{2x}\,dx$$
$$= \int_0^4 \left(2x - \frac{1}{2}x^2\right) dx = \left[x^2 - \frac{x^3}{6}\right]_0^4 = \frac{16}{3}$$

Treating R as type II yields

$$\text{area of } R = \iint_R dA = \int_0^8 \int_{y/2}^{\sqrt{2y}} dx\,dy = \int_0^8 [x]_{x=y/2}^{\sqrt{2y}}\,dy$$
$$= \int_0^8 \left(\sqrt{2y} - \frac{1}{2}y\right) dy = \left[\frac{2\sqrt{2}}{3}y^{3/2} - \frac{y^2}{4}\right]_0^8 = \frac{16}{3}$$



▶ Figure 14.2.14

Question : 9

Region bounded by two surfaces Find the volume of the solid region bounded by the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$ (Figure 13.23).

Solution

The upper surface bounding the solid is $z = 8 - x^2 - y^2$ and the lower surface is $z = x^2 + y^2$. The two surfaces intersect along a curve C. Solving $8 - x^2 - y^2 = x^2 + y^2$, we find that $x^2 + y^2 = 4$. This circle of radius 2 is the projection of C onto the xy-plane (Figure 13.23); it is also the boundary of the region of integration

$$R = \{(x, y): -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}, -2 \le x \le 2\}.$$

Notice that R and the solid are symmetric about the x- and y-axes. Therefore, the volume of the entire solid is four times the volume over that part of R in the first quadrant. The volume of the solid is

$$4\int_0^2 \int_0^{\sqrt{4-x^2}} ((\underbrace{8 - x^2 - y^2}_{g(x,y)}) - (\underbrace{x^2 + y^2}_{f(x,y)}))\, dy\, dx$$

$$= 8\int_0^2 \int_0^{\sqrt{4-x^2}} (4 - x^2 - y^2)\, dy\, dx \quad \text{Simplify the integrand.}$$

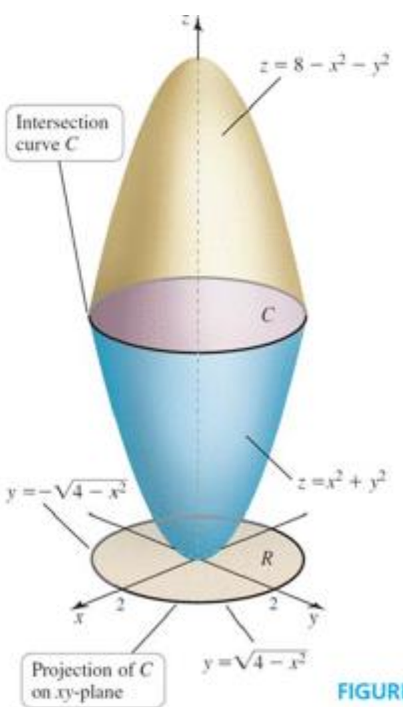
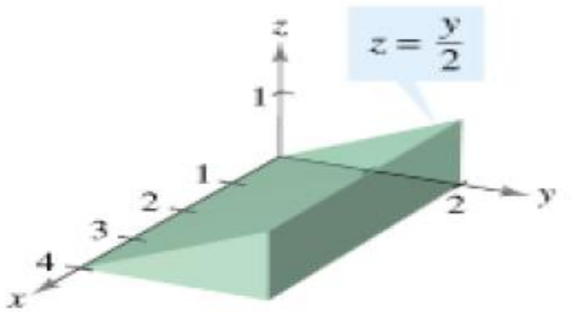


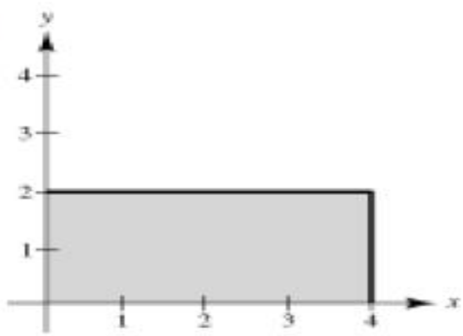
FIGURE 13.23

Question : 10

Use a double integral to find the volume of the indicated solid.



1.



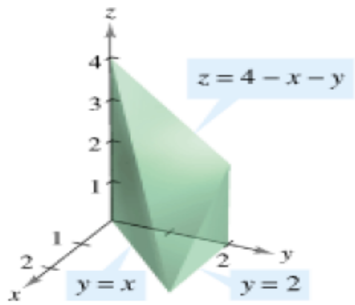
2. $\int_0^4 \int_0^2 \frac{y}{2}\, dy\, dx = \int_0^4 \left[\frac{y^2}{4}\right]_0^2 dx$

3. $= \int_0^4 dx$

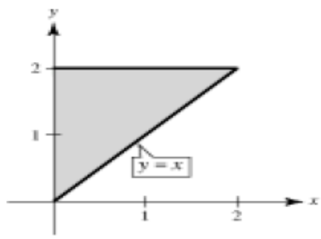
4. $= 4$

Question : 11

Use a double integral to find the volume of the indicated solid.



1.



2. $\int_0^2 \int_0^y (4 - x - y)\, dx\, dy = \int_0^2 \left[4x - \frac{x^2}{2} - xy \right]_0^y dy$

3. $= \int_0^2 \left(4y - \frac{y^2}{2} - y^2 \right) dy$

4. $= \left[2y^2 - \frac{y^3}{6} - \frac{y^3}{3} \right]_0^2$

5. $= 8 - \frac{8}{6} - \frac{8}{3}$

6. $= 4$

Question : 12

Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0, 1] \times [0, 2]$ (Figure 14.1.6).

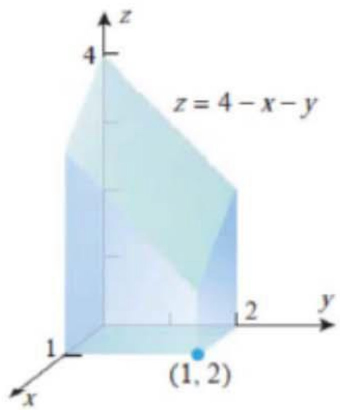


Figure 14.1.6

The volume is the double integral of $z = 4 - x - y$ over R.

$$\int_0^2 \int_0^1 (4 - x - y)\, dx\, dy \quad \text{or} \quad \int_0^1 \int_0^2 (4 - x - y)\, dy\, dx$$

Using the first of these, we obtain

$$\begin{aligned} V &= \iint_R (4 - x - y)\, dA = \int_0^2 \int_0^1 (4 - x - y)\, dx\, dy \\ &= \int_0^2 \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{1} dy = \int_0^2 \left(\frac{7}{2} - y\right) dy \\ &= \left[\frac{7}{2}y - \frac{y^2}{2}\right]_0^2 = 5 \end{aligned}$$

You can check this result by evaluating the second integral