

MATH152 CALCULUS II TUTORIAL – VIII

(07.12.2018)

Question 1:

Evaluate the iterated integral.

$$\int_1^2 \int_0^4 (x^2 - 2y^2 + 1) \, dx \, dy$$

$$1. \int_1^2 \int_0^4 (x^2 - 2y^2 + 1) \, dx \, dy = \int_1^2 \left[\frac{1}{3}x^3 - 2xy^2 + x \right]_0^4 \, dy$$

$$2. \qquad \qquad \qquad = \int_1^2 \left(\frac{64}{3} - 8y^2 + 4 \right) \, dy$$

$$3. \qquad \qquad \qquad = \frac{4}{3} \int_1^2 (19 - 6y^2) \, dy$$

$$4. \qquad \qquad \qquad = \left[\frac{4}{3}(19y - 2y^3) \right]_1^2$$

$$5. \qquad \qquad \qquad = \frac{20}{3}$$

Question 2:

Evaluate the iterated integral.

$$\int_0^1 \int_0^{\sqrt{1-y^2}} (x+y) \, dx \, dy$$

$$1. \int_0^1 \int_0^{\sqrt{1-y^2}} (x+y) \, dx \, dy = \int_0^1 \left[\frac{1}{2}x^2 + xy \right]_0^{\sqrt{1-y^2}} \, dy$$

$$2. \qquad \qquad \qquad = \int_0^1 \left[\frac{1}{2}(1-y^2) + y\sqrt{1-y^2} \right] \, dy$$

$$3. \qquad \qquad \qquad = \left[\frac{1}{2}y - \frac{1}{6}y^3 - \frac{1}{2} \left(\frac{2}{3} \right) (1-y^2)^{3/2} \right]_0^1$$

$$4. \qquad \qquad \qquad = \frac{2}{3}$$

Question 3:

Use an iterated integral to find the area of the region bounded by the graphs of the equations.

$$\sqrt{x} + \sqrt{y} = 2, \quad x = 0, \quad y = 0$$

$$1. \int_0^4 \int_0^{(2-\sqrt{x})^2} dy \, dx = \int_0^4 \left[y \right]_0^{(2-\sqrt{x})^2} \, dx$$

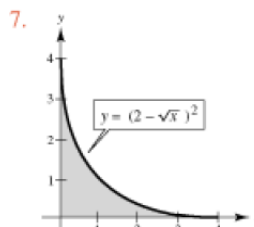
$$2. \qquad \qquad \qquad = \int_0^4 (4 - 4\sqrt{x} + x) \, dx$$

$$3. \qquad \qquad \qquad = \left[4x - \frac{8}{3}x\sqrt{x} + \frac{x^2}{2} \right]_0^4$$

$$4. \qquad \qquad \qquad = \frac{8}{3}$$

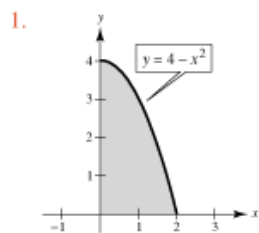
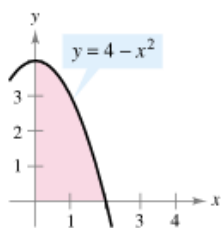
$$5. \int_0^4 \int_0^{(2-\sqrt{y})^2} dx \, dy = \frac{8}{3}$$

6. Integration steps are similar to those above.



Question 4:

Use an iterated integral to find the area of the region.

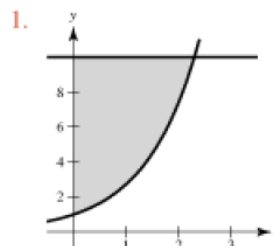


2. $A = \int_0^2 \int_0^{4-x^2} dy dx$
3. $= \int_0^2 [y]_0^{4-x^2} dx$
4. $= \int_0^2 (4 - x^2) dx$
5. $= \left[4x - \frac{x^3}{3} \right]_0^2$
6. $= \frac{16}{3}$
7. $A = \int_0^4 \int_0^{\sqrt{4-y}} dx dy$
8. $= \int_0^4 [x]_0^{\sqrt{4-y}} dy$
9. $= \int_0^4 \sqrt{4-y} dy$
10. $= - \int_0^4 (4-y)^{1/2} (-1) dy$
11. $= \left[-\frac{2}{3} (4-y)^{3/2} \right]_0^4$
12. $= \frac{2}{3} (8)$
13. $= \frac{16}{3}$

Question 5:

Sketch the region R of integration and switch the order of integration.

$$\int_1^{10} \int_0^{\ln y} f(x, y) dx dy$$

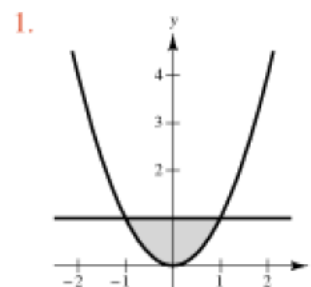


2. $\int_1^{10} \int_0^{\ln y} f(x, y) dx dy, 0 \leq x \leq \ln y, 1 \leq y \leq 10$
3. $= \int_0^{\ln 10} \int_{e^x}^{10} f(x, y) dy dx$

Question 6:

Sketch the region R of integration and switch the order of integration.

$$\int_{-1}^1 \int_{x^2}^1 f(x, y) dy dx$$



2. $\int_{-1}^1 \int_{x^2}^1 f(x, y) dy dx, x^2 \leq y \leq 1, -1 \leq x \leq 1$
3. $= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y) dx dy$

Question : 7

Use an iterated integral to find the area of the region bounded by the graphs of the equations.

$$2x - 3y = 0, \quad x + y = 5, \quad y = 0$$

$$1. \quad A = \int_0^3 \int_0^{2x/3} dy \, dx + \int_3^5 \int_0^{5-x} dy \, dx$$

$$2. \quad = \int_0^3 \left[y \right]_0^{2x/3} dx + \int_3^5 \left[y \right]_0^{5-x} dx$$

$$3. \quad = \int_0^3 \frac{2x}{3} dx + \int_3^5 (5 - x) dx$$

$$4. \quad = \left[\frac{1}{3}x^2 \right]_0^3 + \left[5x - \frac{1}{2}x^2 \right]_3^5$$

$$5. \quad = 5$$

$$6. \quad A = \int_0^2 \int_{3y/2}^{5-y} dx \, dy$$

$$7. \quad = \int_0^2 \left[x \right]_{3y/2}^{5-y} dy$$

$$8. \quad = \int_0^2 \left(5 - y - \frac{3y}{2} \right) dy$$

$$9. \quad = \int_0^2 \left(5 - \frac{5y}{2} \right) dy$$

Question : 8

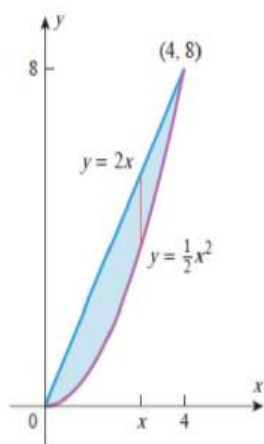
Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

Solution

The region R may be treated equally well as type I (Figure 14.2.14a) or type II (Figure 14.2.14b).

Treating R as type I yields

$$\begin{aligned} \text{area of } R &= \iint_R dA = \int_0^4 \int_{x^2/2}^{2x} dy dx = \int_0^4 [y]_{y=x^2/2}^{2x} dx \\ &= \int_0^4 \left(2x - \frac{1}{2}x^2 \right) dx = \left[x^2 - \frac{x^3}{6} \right]_0^4 = \frac{16}{3} \end{aligned}$$

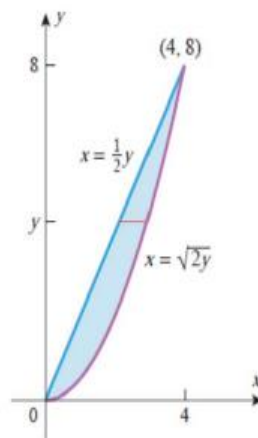


► Figure 14.2.14

(a)

Treating R as type II yields

$$\begin{aligned} \text{area of } R &= \iint_R dA = \int_0^8 \int_{y/2}^{\sqrt{2y}} dx dy = \int_0^8 [x]_{x=y/2}^{\sqrt{2y}} dy \\ &= \int_0^8 \left(\sqrt{2y} - \frac{1}{2}y \right) dy = \left[\frac{2\sqrt{2}}{3} y^{3/2} - \frac{y^2}{4} \right]_0^8 = \frac{16}{3} \end{aligned}$$



(b)

Question : 9

Region bounded by two surfaces Find the volume of the solid region bounded by the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$ (Figure 13.23).

Solution

The upper surface bounding the solid is $z = 8 - x^2 - y^2$ and the lower surface is $z = x^2 + y^2$. The two surfaces intersect along a curve C . Solving $8 - x^2 - y^2 = x^2 + y^2$, we find that $x^2 + y^2 = 4$. This circle of radius 2 is the projection of C onto the xy -plane (Figure 13.23); it is also the boundary of the region of integration

$$R = \{(x, y): -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}, -2 \leq x \leq 2\}.$$

Notice that R and the solid are symmetric about the x - and y -axes. Therefore, the volume of the entire solid is four times the volume over that part of R in the first quadrant. The volume of the solid is

$$\begin{aligned} &4 \int_0^2 \int_0^{\sqrt{4-x^2}} \left(\underbrace{(8 - x^2 - y^2)}_{g(x,y)} - \underbrace{(x^2 + y^2)}_{f(x,y)} \right) dy dx \\ &= 8 \int_0^2 \int_0^{\sqrt{4-x^2}} (4 - x^2 - y^2) dy dx \quad \text{Simplify the integrand.} \end{aligned}$$

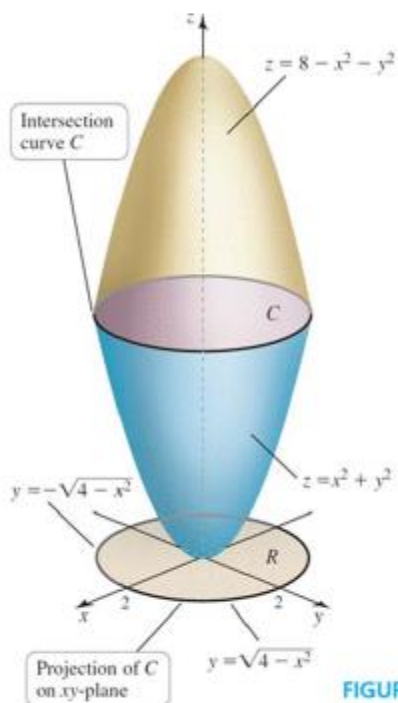
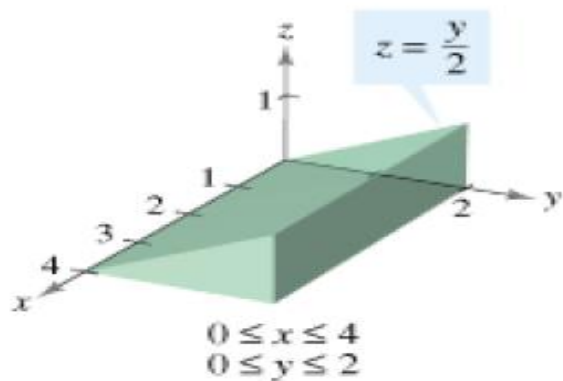


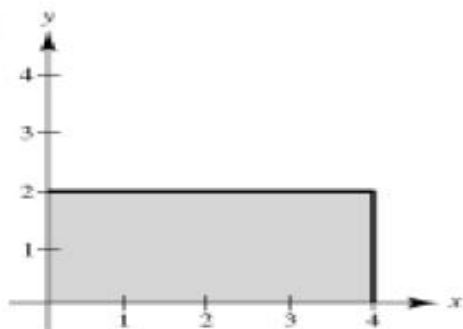
FIGURE 13.23

Question : 10

Use a double integral to find the volume of the indicated solid.



1.



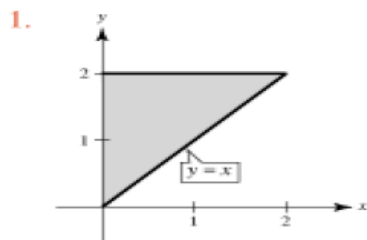
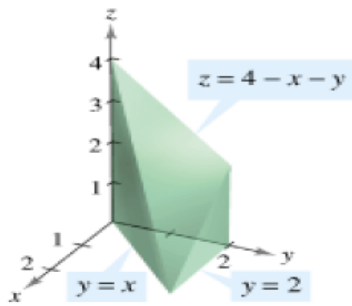
2.
$$\int_0^4 \int_0^2 \frac{y}{2} dy dx = \int_0^4 \left[\frac{y^2}{4} \right]_0^2 dx$$

3.
$$= \int_0^4 dx$$

4.
$$= 4$$

Question : 11

Use a double integral to find the volume of the indicated solid.



2.
$$\int_0^2 \int_0^y (4 - x - y) dx dy = \int_0^2 \left[4x - \frac{x^2}{2} - xy \right]_0^y dy$$

3.
$$= \int_0^2 \left(4y - \frac{y^2}{2} - y^2 \right) dy$$

4.
$$= \left[2y^2 - \frac{y^3}{6} - \frac{y^3}{3} \right]_0^2$$

5.
$$= 8 - \frac{8}{6} - \frac{8}{3}$$

6.
$$= 4$$

Question : 12

Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0, 1] \times [0, 2]$ (Figure 14.1.6).

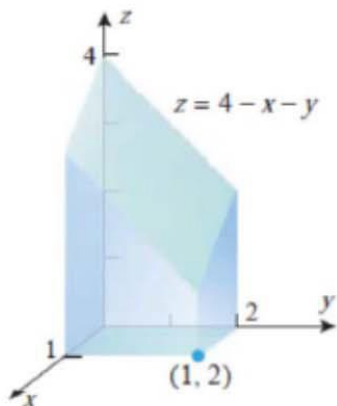


Figure 14.1.6

The volume is the double integral of $z = 4 - x - y$ over R .

$$\int_0^2 \int_0^1 (4 - x - y) dx dy \quad \text{or} \quad \int_0^1 \int_0^2 (4 - x - y) dy dx$$

Using the first of these, we obtain

$$\begin{aligned} V &= \iint_R (4 - x - y) dA = \int_0^2 \int_0^1 (4 - x - y) dx dy \\ &= \int_0^2 \left[4x - \frac{x^2}{2} - xy \right]_{x=0}^1 dy = \int_0^2 \left(\frac{7}{2} - y \right) dy \\ &= \left[\frac{7}{2}y - \frac{y^2}{2} \right]_0^2 = 5 \end{aligned}$$

You can check this result by evaluating the second integral