

## MATH152 CALCULUS II TUTORIAL – VIII

(03.05.2019)

### Question 1:

Evaluate the iterated integral.

$$\int_1^2 \int_0^4 (x^2 - 2y^2 + 1) dx dy$$

1.  $\int_1^2 \int_0^4 (x^2 - 2y^2 + 1) dx dy = \int_1^2 \left[ \frac{1}{3}x^3 - 2xy^2 + x \right]_0^4 dy$

2.  $= \int_1^2 \left( \frac{64}{3} - 8y^2 + 4 \right) dy$

3.  $= \frac{4}{3} \int_1^2 (19 - 6y^2) dy$

4.  $= \left[ \frac{4}{3}(19y - 2y^3) \right]_1^2$

5.  $= \frac{20}{3}$

### Question 2:

Evaluate the iterated integral.

$$\int_0^1 \int_0^{\sqrt{1-y^2}} (x+y) dx dy$$

1.  $\int_0^1 \int_0^{\sqrt{1-y^2}} (x+y) dx dy = \int_0^1 \left[ \frac{1}{2}x^2 + xy \right]_0^{\sqrt{1-y^2}} dy$

2.  $= \int_0^1 \left[ \frac{1}{2}(1-y^2) + y\sqrt{1-y^2} \right] dy$

3.  $= \left[ \frac{1}{2}y - \frac{1}{6}y^3 - \frac{1}{2} \left( \frac{2}{3} \right) (1-y^2)^{3/2} \right]_0^1$

4.  $= \frac{2}{3}$

### Question 3:

Use an iterated integral to find the area of the region bounded by the graphs of the equations.

$$\sqrt{x} + \sqrt{y} = 2, \quad x = 0, \quad y = 0$$

1.  $\int_0^4 \int_0^{(2-\sqrt{x})^2} dy dx = \int_0^4 \left[ y \right]_0^{(2-\sqrt{x})^2} dx$

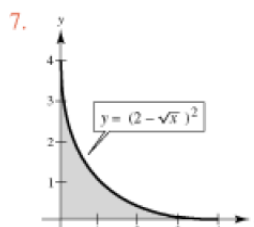
2.  $= \int_0^4 (4 - 4\sqrt{x} + x) dx$

3.  $= \left[ 4x - \frac{8}{3}x\sqrt{x} + \frac{x^2}{2} \right]_0^4$

4.  $= \frac{8}{3}$

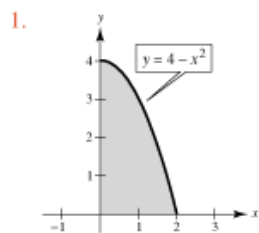
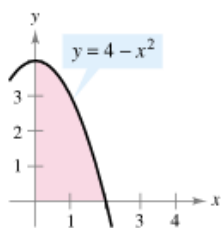
5.  $\int_0^4 \int_0^{(2-\sqrt{y})^2} dx dy = \frac{8}{3}$

6. Integration steps are similar to those above.



**Question 4:**

Use an iterated integral to find the area of the region.

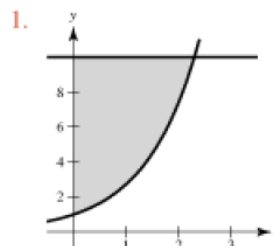


2.  $A = \int_0^2 \int_0^{4-x^2} dy dx$
3.  $= \int_0^2 [y]_0^{4-x^2} dx$
4.  $= \int_0^2 (4 - x^2) dx$
5.  $= \left[ 4x - \frac{x^3}{3} \right]_0^2$
6.  $= \frac{16}{3}$
7.  $A = \int_0^4 \int_0^{\sqrt{4-y}} dx dy$
8.  $= \int_0^4 [x]_0^{\sqrt{4-y}} dy$
9.  $= \int_0^4 \sqrt{4-y} dy$
10.  $= - \int_0^4 (4-y)^{1/2} (-1) dy$
11.  $= \left[ -\frac{2}{3} (4-y)^{3/2} \right]_0^4$
12.  $= \frac{2}{3} (8)$
13.  $= \frac{16}{3}$

**Question 5:**

Sketch the region  $R$  of integration and switch the order of integration.

$$\int_1^{10} \int_0^{\ln y} f(x, y) dx dy$$

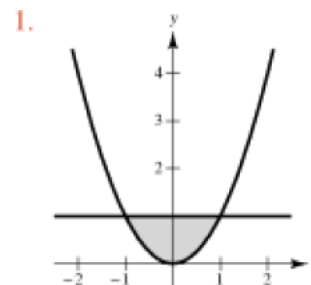


2.  $\int_1^{10} \int_0^{\ln y} f(x, y) dx dy, 0 \leq x \leq \ln y, 1 \leq y \leq 10$
3.  $= \int_0^{\ln 10} \int_{e^x}^{10} f(x, y) dy dx$

**Question 6:**

Sketch the region  $R$  of integration and switch the order of integration.

$$\int_{-1}^1 \int_{x^2}^1 f(x, y) dy dx$$



2.  $\int_{-1}^1 \int_{x^2}^1 f(x, y) dy dx, x^2 \leq y \leq 1, -1 \leq x \leq 1$
3.  $= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y) dx dy$

**Question : 7**

Use an iterated integral to find the area of the region bounded by the graphs of the equations.

$$2x - 3y = 0, \quad x + y = 5, \quad y = 0$$

$$1. A = \int_0^3 \int_0^{2x/3} dy \, dx + \int_3^5 \int_0^{5-x} dy \, dx$$

$$2. = \int_0^3 \left[ y \right]_0^{2x/3} dx + \int_3^5 \left[ y \right]_0^{5-x} dx$$

$$3. = \int_0^3 \frac{2x}{3} dx + \int_3^5 (5 - x) dx$$

$$4. = \left[ \frac{1}{3}x^2 \right]_0^3 + \left[ 5x - \frac{1}{2}x^2 \right]_3^5$$

$$5. = 5$$

$$6. A = \int_0^2 \int_{3y/2}^{5-y} dx \, dy$$

$$7. = \int_0^2 \left[ x \right]_{3y/2}^{5-y} dy$$

$$8. = \int_0^2 \left( 5 - y - \frac{3y}{2} \right) dy$$

$$9. = \int_0^2 \left( 5 - \frac{5y}{2} \right) dy$$

### Question : 8

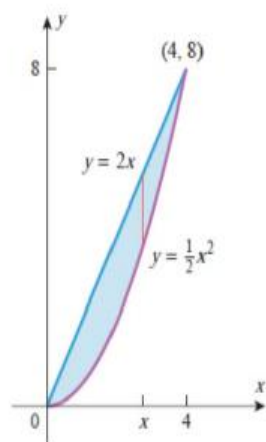
Use a double integral to find the area of the region  $R$  enclosed between the parabola  $y = \frac{1}{2}x^2$  and the line  $y = 2x$ .

#### Solution

The region  $R$  may be treated equally well as type I (Figure 14.2.14a) or type II (Figure 14.2.14b).

Treating  $R$  as type I yields

$$\begin{aligned} \text{area of } R &= \iint_R dA = \int_0^4 \int_{x^2/2}^{2x} dy dx = \int_0^4 [y]_{y=x^2/2}^{2x} dx \\ &= \int_0^4 \left( 2x - \frac{1}{2}x^2 \right) dx = \left[ x^2 - \frac{x^3}{6} \right]_0^4 = \frac{16}{3} \end{aligned}$$

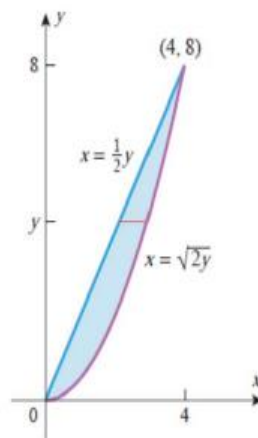


► Figure 14.2.14

(a)

Treating  $R$  as type II yields

$$\begin{aligned} \text{area of } R &= \iint_R dA = \int_0^8 \int_{y/2}^{\sqrt{2y}} dx dy = \int_0^8 [x]_{x=y/2}^{\sqrt{2y}} dy \\ &= \int_0^8 \left( \sqrt{2y} - \frac{1}{2}y \right) dy = \left[ \frac{2\sqrt{2}}{3} y^{3/2} - \frac{y^2}{4} \right]_0^8 = \frac{16}{3} \end{aligned}$$



(b)

**Question : 9**

**Region bounded by two surfaces** Find the volume of the solid region bounded by the paraboloids  $z = x^2 + y^2$  and  $z = 8 - x^2 - y^2$  (Figure 13.23).

**Solution**

The upper surface bounding the solid is  $z = 8 - x^2 - y^2$  and the lower surface is  $z = x^2 + y^2$ . The two surfaces intersect along a curve  $C$ . Solving  $8 - x^2 - y^2 = x^2 + y^2$ , we find that  $x^2 + y^2 = 4$ . This circle of radius 2 is the projection of  $C$  onto the  $xy$ -plane (Figure 13.23); it is also the boundary of the region of integration

$$R = \{(x, y): -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}, -2 \leq x \leq 2\}.$$

Notice that  $R$  and the solid are symmetric about the  $x$ - and  $y$ -axes. Therefore, the volume of the entire solid is four times the volume over that part of  $R$  in the first quadrant. The volume of the solid is

$$\begin{aligned} &4 \int_0^2 \int_0^{\sqrt{4-x^2}} \left( \underbrace{(8 - x^2 - y^2)}_{g(x,y)} - \underbrace{(x^2 + y^2)}_{f(x,y)} \right) dy dx \\ &= 8 \int_0^2 \int_0^{\sqrt{4-x^2}} (4 - x^2 - y^2) dy dx \quad \text{Simplify the integrand.} \end{aligned}$$

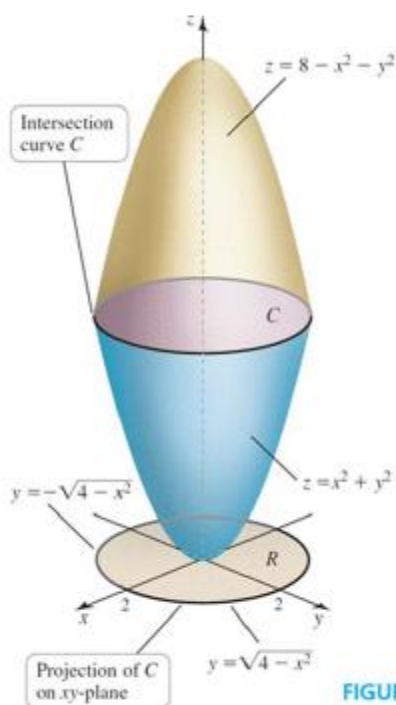
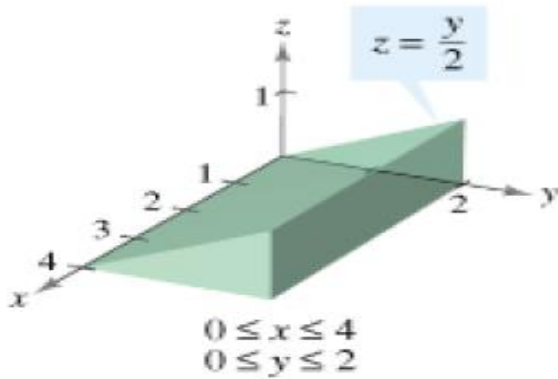


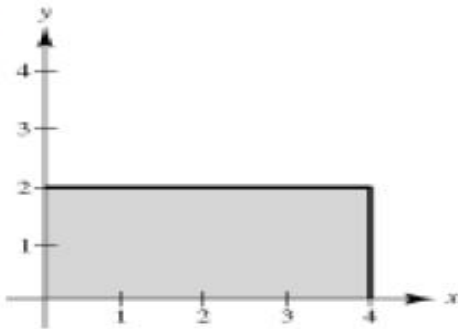
FIGURE 13.23

**Question : 10**

Use a double integral to find the volume of the indicated solid.



1.



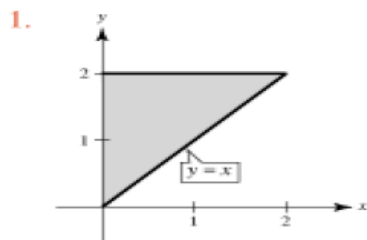
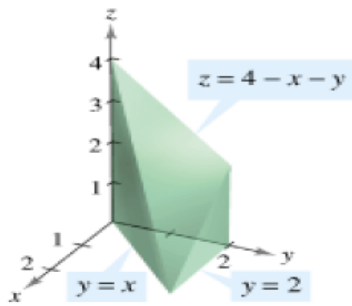
2. 
$$\int_0^4 \int_0^2 \frac{y}{2} dy dx = \int_0^4 \left[ \frac{y^2}{4} \right]_0^2 dx$$

3. 
$$= \int_0^4 dx$$

4. 
$$= 4$$

**Question : 11**

Use a double integral to find the volume of the indicated solid.



2. 
$$\int_0^2 \int_0^y (4 - x - y) dx dy = \int_0^2 \left[ 4x - \frac{x^2}{2} - xy \right]_0^y dy$$

3. 
$$= \int_0^2 \left( 4y - \frac{y^2}{2} - y^2 \right) dy$$

4. 
$$= \left[ 2y^2 - \frac{y^3}{6} - \frac{y^3}{3} \right]_0^2$$

5. 
$$= 8 - \frac{8}{6} - \frac{8}{3}$$

6. 
$$= 4$$

**Question : 12**

Use a double integral to find the volume of the solid that is bounded above by the plane  $z = 4 - x - y$  and below by the rectangle  $R = [0, 1] \times [0, 2]$  (Figure 14.1.6).

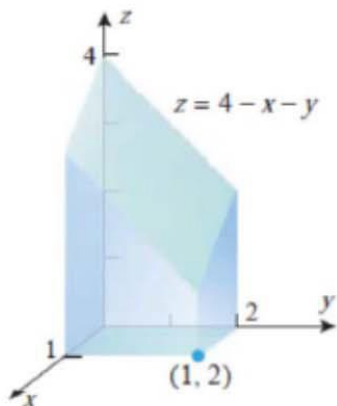


Figure 14.1.6

The volume is the double integral of  $z = 4 - x - y$  over  $R$ .

$$\int_0^2 \int_0^1 (4 - x - y) dx dy \quad \text{or} \quad \int_0^1 \int_0^2 (4 - x - y) dy dx$$

Using the first of these, we obtain

$$\begin{aligned} V &= \iint_R (4 - x - y) dA = \int_0^2 \int_0^1 (4 - x - y) dx dy \\ &= \int_0^2 \left[ 4x - \frac{x^2}{2} - xy \right]_{x=0}^1 dy = \int_0^2 \left( \frac{7}{2} - y \right) dy \\ &= \left[ \frac{7}{2}y - \frac{y^2}{2} \right]_0^2 = 5 \end{aligned}$$

You can check this result by evaluating the second integral