MATH152 CALCULUS II TUTORIAL – VIII

(6.05.2016)

Question 1:

Evaluate

(a) $\int_1^3 \int_2^4 (40 - 2xy)\, dy\, dx$ (b) $\int_2^4 \int_1^3 (40 - 2xy)\, dx\, dy$

Solution

$$\int_1^3 \int_2^4 (40 - 2xy)\, dy\, dx = \int_1^3 \left[\int_2^4 (40 - 2xy)\, dy\right] dx$$
$$= \int_1^3 (40y - xy^2)\Big]_{y=2}^{4}\, dx$$
$$= \int_1^3 [(160 - 16x) - (80 - 4x)]\, dx$$
$$= \int_1^3 (80 - 12x)\, dx$$
$$= (80x - 6x^2)\Big]_1^3 = 112$$

$$\int_2^4 \int_1^3 (40 - 2xy)\, dx\, dy = \int_2^4 \left[\int_1^3 (40 - 2xy)\, dx\right] dy$$
$$= \int_2^4 (40x - x^2y)\Big]_{x=1}^{3}\, dy$$
$$= \int_2^4 [(120 - 9y) - (40 - y)]\, dy$$
$$= \int_2^4 (80 - 8y)\, dy$$
$$= (80y - 4y^2)\Big]_2^4 = 112 \blacktriangleleft$$

Question 2 :

Evaluate the iterated integral.

$$\int_0^{\pi} \int_0^{\sin x} (1 + \cos x)\, dy\, dx$$

1. $\int_0^{\pi} \int_0^{\sin x} (1 + \cos x)\, dy\, dx = \int_0^{\pi} \Big[(y + y\cos x)\Big]_0^{\sin x} dx$
2. $= \int_0^{\pi} [\sin x + \sin x \cos x]\, dx$
3. $= \left[-\cos x + \frac{1}{2}\sin^2 x\right]_0^{\pi}$
4. $= 1 + 1$
5. $= 2$

Question 3 :

Choosing a convenient order of integration Evaluate $\iint_R ye^{xy}\, dA$, where $R = \{(x, y): 0 \le x \le 1, 0 \le y \le \ln 2\}$.

Solution

The iterated integral $\int_0^1 \int_0^{\ln 2} ye^{xy}\, dy\, dx$ requires first integrating ye^{xy} with respect to y, which entails integration by parts. An easier approach is to integrate first with respect to x:

$$\int_0^{\ln 2} \int_0^1 ye^{xy}\, dx\, dy = \int_0^{\ln 2} (e^{xy})\Big|_0^1\, dy$$ Evaluate the inner integral with respect to x.

$$= \int_0^{\ln 2} (e^y - 1)\, dy$$ Simplify.

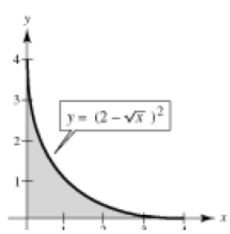
$$= (e^y - y)\Big|_0^{\ln 2}$$ Evaluate the outer integral with respect to y.

$$= 1 - \ln 2$$ Simplify.

Question 4:

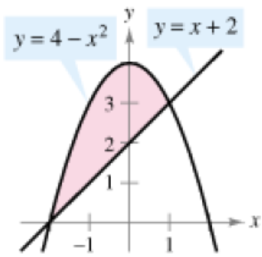
Use an iterated integral to find the area of the region bounded by the graphs of the equations.

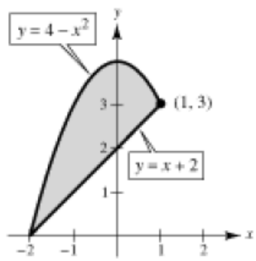
$\sqrt{x} + \sqrt{y} = 2, \quad x = 0, \quad y = 0$

1. $\int_0^4 \int_0^{(2-\sqrt{x})^2} dy\, dx = \int_0^4 \Big[y\Big]_0^{(2-\sqrt{x})^2} dx$
2. $= \int_0^4 (4 - 4\sqrt{x} + x)\, dx$
3. $= \left[4x - \frac{8}{3}x\sqrt{x} + \frac{x^2}{2}\right]_0^4$
4. $= \frac{8}{3}$
5. $\int_0^4 \int_0^{(2-\sqrt{y})^2} dx\, dy = \frac{8}{3}$
6. Integration steps are similar to those above.
7. 

Question 5:

Use an iterated integral to find the area of the region.



1. 

2. $A = \int_{-2}^{1}\int_{x+2}^{4-x^2} dy\,dx$

3. $= \int_{-2}^{1}\Big[y\Big]_{x+2}^{4-x^2} dx$

4. $= \int_{-2}^{1}(4 - x^2 - x - 2)\,dx$

5. $= \int_{-2}^{1}(2 - x - x^2)\,dx$

6. $= \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-2}^{1}$

7. $= \frac{9}{2}$

8. $A = \int_{0}^{3}\int_{-\sqrt{4-y}}^{y-2} dx\,dy + 2\int_{3}^{4}\int_{0}^{\sqrt{4-y}} dx\,dy$

9. $= \int_{0}^{3}\Big[x\Big]_{-\sqrt{4-y}}^{y-2} dy + 2\int_{3}^{4}\Big[x\Big]_{0}^{\sqrt{4-y}} dy$

10. $= \int_{0}^{3}\left(y - 2 + \sqrt{4-y}\right)dy + 2\int_{3}^{4}\sqrt{4-y}\,dy$

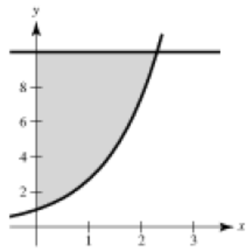
11. $= \left[\frac{1}{2}y^2 - 2y - \frac{2}{3}(4-y)^{3/2}\right]_{0}^{3} - \left[\frac{4}{3}(4-y)^{3/2}\right]_{3}^{4}$

12. $= \frac{9}{2}$

Question 6:

Sketch the region R of integration and switch the order of integration.

$$\int_{1}^{10}\int_{0}^{\ln y} f(x, y)\,dx\,dy$$

1. 

2. $\int_{1}^{10}\int_{0}^{\ln y} f(x, y)\,dx\,dy,\ 0 \le x \le \ln y, 1 \le y \le 10$

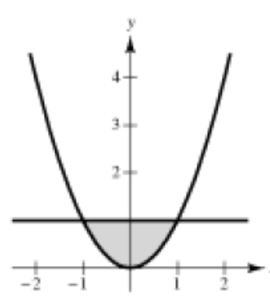
3. $= \int_{0}^{\ln 10}\int_{e^x}^{10} f(x, y)\,dy\,dx$

Question 7:

Sketch the region R of integration and switch the o

of integration.

$$\int_{-1}^{1}\int_{x^2}^{1} f(x, y)\,dy\,dx$$

1. 

2. $\int_{-1}^{1}\int_{x^2}^{1} f(x, y)\,dy\,dx,\ x^2 \le y \le 1, -1 \le x \le 1$

3. $= \int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\,dx\,dy$

Question : 8

Use an iterated integral to find the area of the region bounded by the graphs of the equations.

$2x - 3y = 0, \quad x + y = 5, \quad y = 0$

1. $A = \int_0^3 \int_0^{2x/3} dy\, dx + \int_3^5 \int_0^{5-x} dy\, dx$
2. $= \int_0^3 \Big[y \Big]_0^{2x/3} dx + \int_3^5 \Big[y \Big]_0^{5-x} dx$
3. $= \int_0^3 \frac{2x}{3}\, dx + \int_3^5 (5 - x)\, dx$
4. $= \left[\frac{1}{3}x^2 \right]_0^3 + \left[5x - \frac{1}{2}x^2 \right]_3^5$
5. $= 5$
6. $A = \int_0^2 \int_{3y/2}^{5-y} dx\, dy$
7. $= \int_0^2 \Big[x \Big]_{3y/2}^{5-y} dy$
8. $= \int_0^2 \left(5 - y - \frac{3y}{2} \right) dy$
9. $= \int_0^2 \left(5 - \frac{5y}{2} \right) dy$

Question 9

Area of a plane region Find the area of the region R bounded by $y = x^2$, $y = -x + 12$, and $y = 4x + 12$ (Figure 13.26).

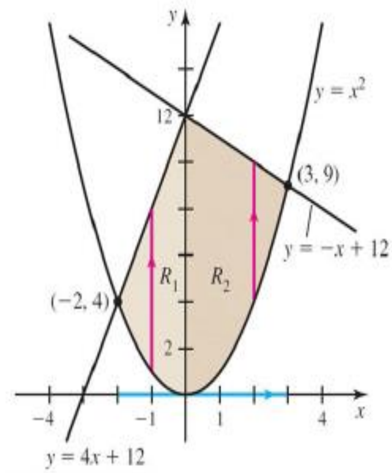


FIGURE 13.26

Solution

The region R in its entirety is bounded neither above and below by two curves, nor on the left and right by two curves. However, when decomposed along the y-axis, R may be viewed as two regions R_1 and R_2 that are each bounded above and below by a pair of curves. Notice that the parabola $y = x^2$ and the line $y = -x + 12$ intersect in the first quadrant at the point $(3, 9)$, while the parabola and the line $y = 4x + 12$ intersect in the second quadrant at the point $(-2, 4)$.

To find the area of R, we integrate the function $f(x, y) = 1$ over R_1 and R_2; the area is

$$\iint_{R_1} 1\,dA + \iint_{R_2} 1\,dA \qquad \text{Decompose region.}$$

$$= \int_{-2}^{0}\int_{x^2}^{4x+12} 1\,dy\,dx + \int_{0}^{3}\int_{x^2}^{-x+12} 1\,dy\,dx \qquad \text{Convert to iterated integrals.}$$

$$= \int_{-2}^{0} (4x + 12 - x^2)\,dx + \int_{0}^{3} (-x + 12 - x^2)\,dx \qquad \text{Evaluate the inner integrals.}$$

$$= \left(2x^2 + 12x - \frac{x^3}{3}\right)\Bigg|_{-2}^{0} + \left(-\frac{x^2}{2} + 12x - \frac{x^3}{3}\right)\Bigg|_{0}^{3} \qquad \text{Evaluate the outer integrals.}$$

$$= \frac{40}{3} + \frac{45}{2} = \frac{215}{6}. \qquad \text{Simplify.}$$

Question : 10

Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$..

Solution

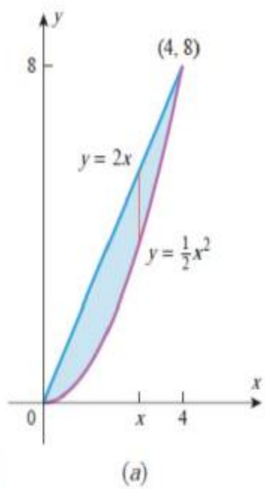
The region R may be treated equally well as type I (Figure 14.2.14a) or type II (Figure 14.2.14b).

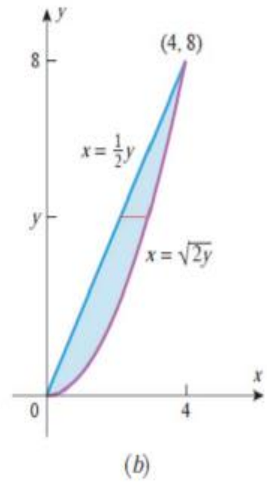
Treating R as type I yields

$$\text{area of } R = \iint_R dA = \int_0^4 \int_{x^2/2}^{2x} dy\,dx = \int_0^4 [y]_{y=x^2/2}^{2x}\,dx$$
$$= \int_0^4 \left(2x - \frac{1}{2}x^2\right) dx = \left[x^2 - \frac{x^3}{6}\right]_0^4 = \frac{16}{3}$$

Treating R as type II yields

$$\text{area of } R = \iint_R dA = \int_0^8 \int_{y/2}^{\sqrt{2y}} dx\,dy = \int_0^8 [x]_{x=y/2}^{\sqrt{2y}}\,dy$$
$$= \int_0^8 \left(\sqrt{2y} - \frac{1}{2}y\right) dy = \left[\frac{2\sqrt{2}}{3}y^{3/2} - \frac{y^2}{4}\right]_0^8 = \frac{16}{3}$$





▶ Figure 14.2.14

Question : 11

Region bounded by two surfaces Find the volume of the solid region bounded by the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$ (Figure 13.23).

Solution

The upper surface bounding the solid is $z = 8 - x^2 - y^2$ and the lower surface is $z = x^2 + y^2$. The two surfaces intersect along a curve C. Solving $8 - x^2 - y^2 = x^2 + y^2$, we find that $x^2 + y^2 = 4$. This circle of radius 2 is the projection of C onto the xy-plane (Figure 13.23); it is also the boundary of the region of integration

$$R = \{(x, y): -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}, -2 \le x \le 2\}.$$

Notice that R and the solid are symmetric about the x- and y-axes. Therefore, the volume of the entire solid is four times the volume over that part of R in the first quadrant. The volume of the solid is

$$4\int_0^2 \int_0^{\sqrt{4-x^2}} ((\underbrace{8 - x^2 - y^2}_{g(x,y)}) - (\underbrace{x^2 + y^2}_{f(x,y)}))\, dy\, dx$$

$$= 8\int_0^2 \int_0^{\sqrt{4-x^2}} (4 - x^2 - y^2)\, dy\, dx \quad \text{Simplify the integrand.}$$

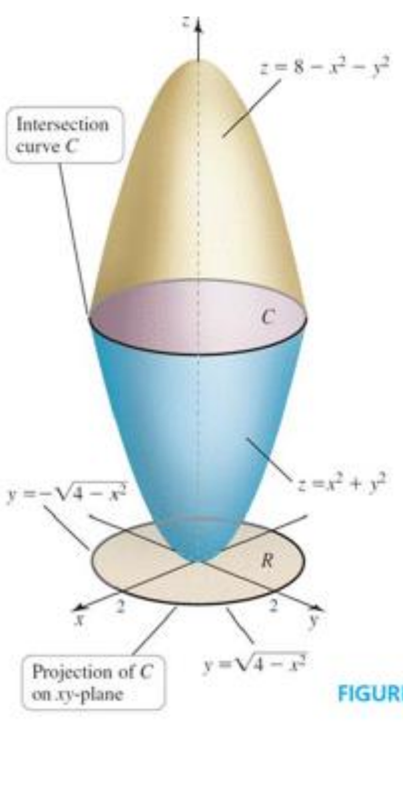
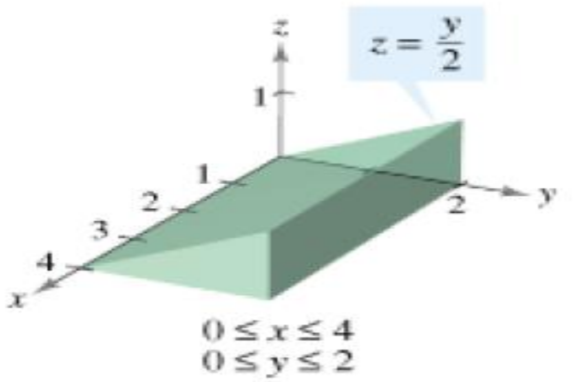


FIGURE 13.23

Question : 12

Use a double integral to find the volume of the indicated solid.



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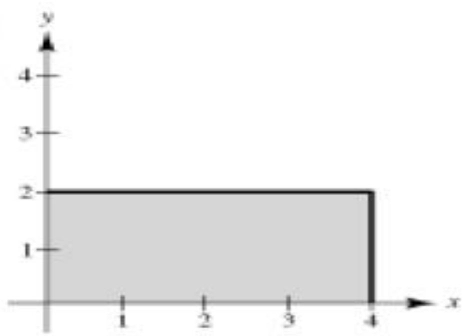


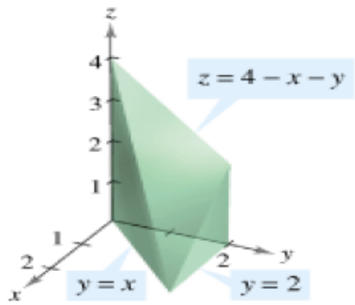
2. $\int_0^4 \int_0^2 \frac{y}{2}\, dy\, dx = \int_0^4 \left[\frac{y^2}{4}\right]_0^2 dx$

3. $= \int_0^4 dx$

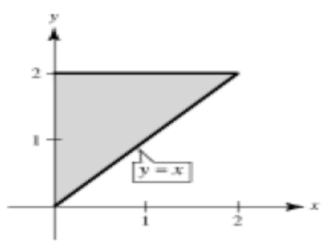
4. $= 4$

Question : 13

Use a double integral to find the volume of the indicated solid.



1.



2. $\int_0^2 \int_0^y (4 - x - y)\, dx\, dy = \int_0^2 \left[4x - \frac{x^2}{2} - xy \right]_0^y dy$

3. $= \int_0^2 \left(4y - \frac{y^2}{2} - y^2 \right) dy$

4. $= \left[2y^2 - \frac{y^3}{6} - \frac{y^3}{3} \right]_0^2$

5. $= 8 - \frac{8}{6} - \frac{8}{3}$

6. $= 4$

Question : 14

Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0, 1] \times [0, 2]$ (Figure 14.1.6).

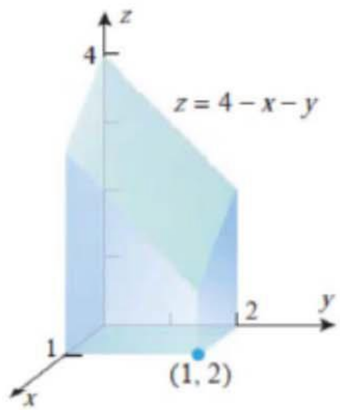


Figure 14.1.6

The volume is the double integral of $z = 4 - x - y$ over R.

$$\int_0^2 \int_0^1 (4 - x - y)\, dx\, dy \quad \text{or} \quad \int_0^1 \int_0^2 (4 - x - y)\, dy\, dx$$

Using the first of these, we obtain

$$V = \iint_R (4 - x - y)\, dA = \int_0^2 \int_0^1 (4 - x - y)\, dx\, dy$$
$$= \int_0^2 \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^1 dy = \int_0^2 \left(\frac{7}{2} - y\right) dy$$
$$= \left[\frac{7}{2}y - \frac{y^2}{2}\right]_0^2 = 5$$

You can check this result by evaluating the second integral