MATH152 CALCULUS II TUTORIAL – VIII

(05.04.2018)

Question 1:

Use the gradient to find the directional derivative of the function at P in the direction of Q.

$f(x, y) = e^{-x} \cos y, \quad P(0, 0), \; Q(2, 1)$

1. $\overrightarrow{PQ} = 2\mathbf{i} + \mathbf{j}$
2. $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$
3. $\nabla f(x, y) = -e^{-x} \cos y\mathbf{i} - e^{-x} \sin y\mathbf{j}$
4. $\nabla f(0, 0) = -\mathbf{i}$
5. $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$
6. $\quad = -\frac{2}{\sqrt{5}}$
7. $\quad = -\frac{2\sqrt{5}}{5}$

Question 2 :

Find the directional derivative of the function at P in the direction of $\mathbf{v}$.

$f(x, y, z) = xy + yz + xz, \quad P(1, 1, 1), \; \mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

1. $\nabla f(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$
2. $\nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
3. $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$
4. $\quad = \frac{\sqrt{6}}{3}\mathbf{i} + \frac{\sqrt{6}}{6}\mathbf{j} - \frac{\sqrt{6}}{6}\mathbf{k}$
5. $D_{\mathbf{u}} f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \mathbf{u}$
6. $\quad = \frac{2\sqrt{6}}{3}$

Question 3 :

Find the gradient of the function and the maximum value of the directional derivative at the given point.

Function	*Point*
$g(x, y) = \ln \sqrt[3]{x^2 + y^2}$	$(1, 2)$

1. $g(x, y) = \ln \sqrt[3]{x^2 + y^2}$
2. $\quad = \frac{1}{3} \ln(x^2 + y^2)$
3. $\nabla g(x, y) = \frac{1}{3}\left[\frac{2x}{x^2 + y^2}\mathbf{i} + \frac{2y}{x^2 + y^2}\mathbf{j}\right]$
4. $\nabla g(1, 2) = \frac{1}{3}\left(\frac{2}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right)$
5. $\quad = \frac{2}{15}(\mathbf{i} + 2\mathbf{j})$
6. $\|\nabla g(1, 2)\| = \frac{2\sqrt{5}}{15}$

Question 4:

Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

(a) increases most rapidly at the point $(1, 1)$.

(b) decreases most rapidly at $(1, 1)$.

(c) What are the directions of zero change in f at $(1, 1)$?

Solution

(a) The function increases most rapidly in the direction of ∇f at $(1, 1)$. The gradient there is

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

(b) The function decreases most rapidly in the direction of $-\nabla f$ at $(1, 1)$, which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

(c) The directions of zero change at (1, 1) are the directions orthogonal to ∇f:

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

See Figure 14.29.

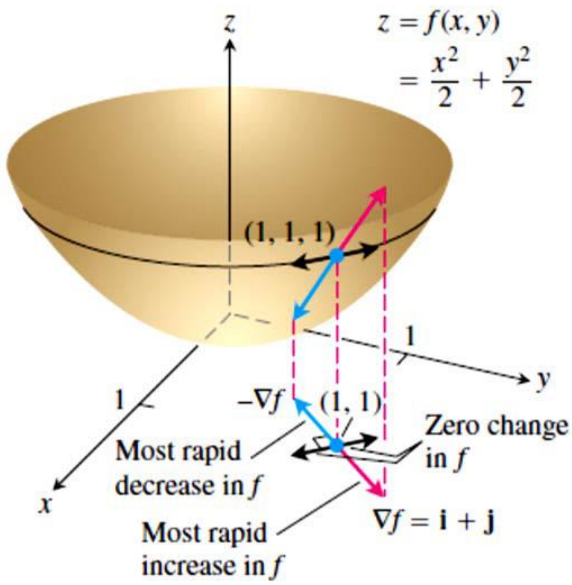


FIGURE 14.29 The direction in which $f(x, y)$ increases most rapidly at (1, 1) is the direction of $\nabla f|_{(1,1)} = \mathbf{i} + \mathbf{j}$. It corresponds to the direction of steepest ascent on the surface at (1, 1, 1) (Example 3).

Question 5:

Find a unit normal vector to the surface at the given point. [*Hint:* Normalize the gradient vector $\nabla F(x, y, z)$.]

Surface	*Point*
$z = \sqrt{x^2 + y^2}$	(3, 4, 5)

1. $F(x, y, z) = \sqrt{x^2 + y^2} - z$
2. $\nabla F(x, y, z) = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j} - \mathbf{k}$
3. $\nabla F(3, 4, 5) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}$
4. $\mathbf{n} = \frac{\nabla F}{\|\nabla F\|}$
5. $= \frac{5}{5\sqrt{2}}\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}\right)$
6. $= \frac{1}{5\sqrt{2}}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$
7. $= \frac{\sqrt{2}}{10}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$

Question : 6

Find an equation of the tangent plane to the surface at the given point.

$x^2 + 4y^2 + z^2 = 36, \quad (2, -2, 4)$

1. $F(x, y, z) = x^2 + 4y^2 + z^2 - 36$
2. $F_x(x, y, z) = 2x$
3. $F_y(x, y, z) = 8y$
4. $F_z(x, y, z) = 2z$
5. $F_x(2, -2, 4) = 4$
6. $F_y(2, -2, 4) = -16$
7. $F_z(2, -2, 4) = 8$
8. $4(x - 2) - 16(y + 2) + 8(z - 4) = 0$
9. $(x - 2) - 4(y + 2) + 2(z - 4) = 0$
10. $x - 4y + 2z = 18$

Question 7

Find an equation of the tangent plane and find symmetric equations of the normal line to the surface at the given point.

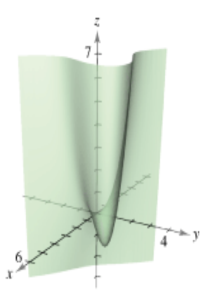
$x^2 + y^2 + z = 9, \quad (1, 2, 4)$

1. $F(x, y, z) = x^2 + y^2 + z - 9$
2. $F_x(x, y, z) = 2x$
3. $F_y(x, y, z) = 2y$
4. $F_z(x, y, z) = 1$
5. $F_x(1, 2, 4) = 2$
6. $F_y(1, 2, 4) = 4$
7. $F_z(1, 2, 4) = 1$
8. Direction numbers: 2, 4, 1
9. Plane: $2(x - 1) + 4(y - 2) + (z - 4) = 0$
10. $2x + 4y + z = 14$
11. Line: $\dfrac{x-1}{2} = \dfrac{y-2}{4} = \dfrac{z-4}{1}$

Question : 8

Examine the function for relative extrema and saddle points.

$f(x, y) = x^3 - 3xy + y^3$



1. $\left.\begin{aligned} f_x &= 3(x^2 - y) = 0 \\ f_y &= 3(-x + y^2) = 0 \end{aligned}\right\}$ Solving by substitution yields two critical points (0, 0) and (1, 1).
2. $f_{xx} = 6x$
3. $f_{yy} = 6y$
4. $f_{xy} = -3$
5. At the critical point (0, 0), $f_{xx} f_{yy} - (f_{xy})^2 < 0$.
6. Therefore, (0, 0, 0) is a saddle point.
7. At the critical point (1, 1), $f_{xx} = 6 > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$.
8. Therefore, (1, 1, −1) is a relative minimum.

Question : 9

Examine the function for relative extrema and saddle points.

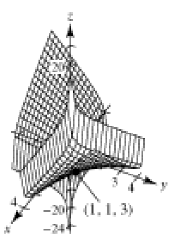
$h(x, y) = x^2 - y^2 - 2x - 4y - 4$

1. $h_x = 2x - 2$
2. $= 2(x - 1)$
3. $= 0$ when $x = 1$.
4. $h_y = -2y - 4$
5. $= -2(y + 2)$
6. $= 0$ when $y = -2$.
7. $h_{xx} = 2$
8. $h_{yy} = -2$
9. $h_{xy} = 0$
10. At the critical point $(1, -2)$, $h_{xx} h_{yy} - (h_{xy})^2 < 0$.
11. Therefore, $(1, -2, -1)$ is a saddle point.

Question 10

Examine the function for relative extrema. Use a computer algebra system to graph the function and confirm your results.

$f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$

1. $f_x = y - \frac{1}{x^2}$
2. $= 0$
3. $x^2y = 1$
4. $f_y = x - \frac{1}{y^2}$
5. $= 0$
6. $xy^2 = 1$
7. Thus, $x^2y = xy^2$ or
8. $x = y$
9. and substitution yields the critical point $(1, 1)$.
10. $f_{xx} = \frac{2}{x^3}$
11. $f_{xy} = 1$
12. $f_{yy} = \frac{2}{y^3}$
13. At the critical point $(1, 1)$, $f_{xx} = 2 > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = 3 > 0$.
14. Thus, $(1, 1, 3)$ is a relative minimum.
15. 

Question 11

Find all critical points of the following function.

$$f(x,y) = \frac{1}{3}x^3 - 7y^3 - 5x + 21y - 7$$

An interior point (a,b) in the domain of f is a critical point of f if either $f_x(a,b) = f_y(a,b) = 0$, or one (or both) of f_x or f_y does not exist at (a,b).

Find the partial derivatives f_x and f_y.

$f_x = x^2 - 5$

$f_y = -21y^2 + 21$

Set each derivative equal to zero and solve the system of equations for x and y.

$$\begin{aligned} x^2 - 5 &= 0 \\ -21y^2 + 21 &= 0 \end{aligned}$$

Note that there are no y-terms in the equation for f_x, and no x-terms in the equation for f_y. Also note that there are two solutions for each equation. This means that there are four ordered pairs in total (every possible combination of each solution for x and y).

Solve the equation $f_x = 0$ for x.

$$\begin{aligned} x^2 - 5 &= 0 \\ x &= \pm\sqrt{5} \end{aligned}$$

Solve the equation $f_y = 0$ for y.

$$\begin{aligned} -21y^2 + 21 &= 0 \\ y &= \pm 1 \end{aligned}$$

Since x can be $\pm\sqrt{5}$ and y can be ± 1, there are four critical points for $f(x,y)$ at $(\sqrt{5},1)$, $(\sqrt{5},-1)$, $(-\sqrt{5},1)$, and $(-\sqrt{5},-1)$.

Question 12

Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4$$

Solution

Since

$$\begin{aligned} f_x(x, y) &= 4y - 4x^3 \\ f_y(x, y) &= 4x - 4y^3 \end{aligned} \tag{1}$$

the critical points of f have coordinates satisfying the equations

$$\begin{aligned} 4y - 4x^3 &= 0 \\ 4x - 4y^3 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} y &= x^3 \\ x &= y^3 \end{aligned} \tag{2}$$

Substituting the top equation in the bottom yields $x = (x^3)^3$ or, equivalently, $x^9 - x = 0$ or $x(x^8 - 1) = 0$, which has solutions $x = 0$, $x = 1$, $x = -1$. Substituting these values in the top equation of (2), we obtain the corresponding y-values $y = 0$, $y = 1$, $y = -1$. Thus,

the critical points of f are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

From (1),

$$f_{xx}(x, y) = -12x^2, \quad f_{yy}(x, y) = -12y^2, \quad f_{xy}(x, y) = 4$$

which yields the following table:

CRITICAL POINT (x_0, y_0)	$f_{xx}(x_0, y_0)$	$f_{yy}(x_0, y_0)$	$f_{xy}(x_0, y_0)$	$D = f_{xx}f_{yy} - f_{xy}^2$
(0, 0)	0	0	4	−16
(1, 1)	−12	−12	4	128
(−1, −1)	−12	−12	4	128

At the points $(1, 1)$ and $(-1, -1)$, we have $D > 0$ and $f_{xx} < 0$, so relative maxima occur at these critical points. At $(0, 0)$ there is a saddle point since $D < 0$.