

MATH152 CALCULUS II TUTORIAL – VIII

(16.11.2018)

Question 1:

Use the gradient to find the directional derivative of the function at P in the direction of Q .

$$f(x, y) = e^{-x} \cos y, \quad P(0, 0), \quad Q(2, 1)$$

1. $\overrightarrow{PQ} = 2\mathbf{i} + \mathbf{j}$
2. $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$
3. $\nabla f(x, y) = -e^{-x} \cos y \mathbf{i} - e^{-x} \sin y \mathbf{j}$
4. $\nabla f(0, 0) = -\mathbf{i}$
5. $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$
6. $= -\frac{2}{\sqrt{5}}$
7. $= -\frac{2\sqrt{5}}{5}$

Question 2:

Find the directional derivative of the function at P in the direction of \mathbf{v} .

$$f(x, y, z) = xy + yz + xz, \quad P(1, 1, 1), \quad \mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

1. $\nabla f(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$
2. $\nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
3. $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$
4. $= \frac{\sqrt{6}}{3}\mathbf{i} + \frac{\sqrt{6}}{6}\mathbf{j} - \frac{\sqrt{6}}{6}\mathbf{k}$
5. $D_{\mathbf{u}}f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \mathbf{u}$
6. $= \frac{2\sqrt{6}}{3}$

Question 3:

Find the gradient of the function and the maximum value of the directional derivative at the given point.

<u>Function</u>	<u>Point</u>
$g(x, y) = \ln \sqrt[3]{x^2 + y^2}$	$(1, 2)$

1. $g(x, y) = \ln \sqrt[3]{x^2 + y^2}$
2. $= \frac{1}{3} \ln(x^2 + y^2)$
3. $\nabla g(x, y) = \frac{1}{3} \left[\frac{2x}{x^2 + y^2} \mathbf{i} + \frac{2y}{x^2 + y^2} \mathbf{j} \right]$
4. $\nabla g(1, 2) = \frac{1}{3} \left(\frac{2}{5} \mathbf{i} + \frac{4}{5} \mathbf{j} \right)$
5. $= \frac{2}{15} (\mathbf{i} + 2\mathbf{j})$
6. $\|\nabla g(1, 2)\| = \frac{2\sqrt{5}}{15}$

Question 4:

Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

- (a) increases most rapidly at the point $(1, 1)$.
- (b) decreases most rapidly at $(1, 1)$.
- (c) What are the directions of zero change in f at $(1, 1)$?

Solution

- (a) The function increases most rapidly in the direction of ∇f at $(1, 1)$. The gradient there is

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{\|\mathbf{i} + \mathbf{j}\|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

- (b) The function decreases most rapidly in the direction of $-\nabla f$ at $(1, 1)$, which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

(c) The directions of zero change at $(1, 1)$ are the directions orthogonal to ∇f :

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

See Figure 14.29.

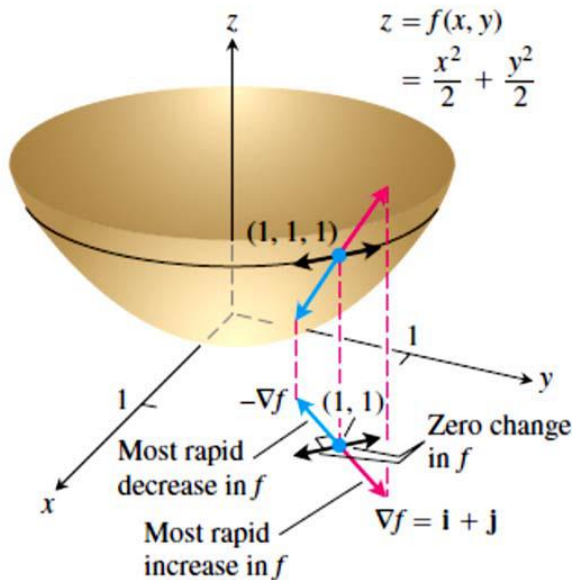


FIGURE 14.29 The direction in which $f(x, y)$ increases most rapidly at $(1, 1)$ is the direction of $\nabla f|_{(1,1)} = \mathbf{i} + \mathbf{j}$. It corresponds to the direction of steepest ascent on the surface at $(1, 1, 1)$ (Example 3).

Question 5:

Find a unit normal vector to the surface at the given point. [Hint: Normalize the gradient vector $\nabla F(x, y, z)$.]

Surface	Point
$z = \sqrt{x^2 + y^2}$	$(3, 4, 5)$

- $F(x, y, z) = \sqrt{x^2 + y^2} - z$
- $\nabla F(x, y, z) = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j} - \mathbf{k}$
- $\nabla F(3, 4, 5) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}$
- $\mathbf{n} = \frac{\nabla F}{\|\nabla F\|}$
- $= \frac{5}{5\sqrt{2}}\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}\right)$
- $= \frac{1}{5\sqrt{2}}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$
- $= \frac{\sqrt{2}}{10}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$

Question : 6

Find an equation of the tangent plane to the surface at the given point.

$$x^2 + 4y^2 + z^2 = 36, \quad (2, -2, 4)$$

- $F(x, y, z) = x^2 + 4y^2 + z^2 - 36$
- $F_x(x, y, z) = 2x$
- $F_y(x, y, z) = 8y$
- $F_z(x, y, z) = 2z$
- $F_x(2, -2, 4) = 4$
- $F_y(2, -2, 4) = -16$
- $F_z(2, -2, 4) = 8$
- $4(x - 2) - 16(y + 2) + 8(z - 4) = 0$
- $(x - 2) - 4(y + 2) + 2(z - 4) = 0$
- $x - 4y + 2z = 18$

Question 7

Find an equation of the tangent plane and find symmetric equations of the normal line to the surface at the given point.

$$x^2 + y^2 + z = 9, \quad (1, 2, 4)$$

1. $F(x, y, z) = x^2 + y^2 + z - 9$
2. $F_x(x, y, z) = 2x$
3. $F_y(x, y, z) = 2y$
4. $F_z(x, y, z) = 1$
5. $F_x(1, 2, 4) = 2$
6. $F_y(1, 2, 4) = 4$
7. $F_z(1, 2, 4) = 1$
8. Direction numbers: 2, 4, 1
9. Plane: $2(x - 1) + 4(y - 2) + (z - 4) = 0$
10. $2x + 4y + z = 14$
11. Line: $\frac{x - 1}{2} = \frac{y - 2}{4} = \frac{z - 4}{1}$

Question : 9

Examine the function for relative extrema and saddle points.

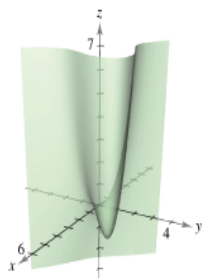
$$h(x, y) = x^2 - y^2 - 2x - 4y - 4$$

1. $h_x = 2x - 2$
2. $= 2(x - 1)$
3. $= 0$ when $x = 1$.
4. $h_y = -2y - 4$
5. $= -2(y + 2)$
6. $= 0$ when $y = -2$.
7. $h_{xx} = 2$
8. $h_{yy} = -2$
9. $h_{xy} = 0$
10. At the critical point $(1, -2)$, $h_{xx} h_{yy} - (h_{xy})^2 < 0$.
11. Therefore, $(1, -2, -1)$ is a saddle point.

Question : 8

Examine the function for relative extrema and saddle points.

$$f(x, y) = x^3 - 3xy + y^3$$



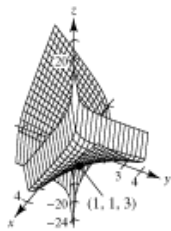
1. $\left. \begin{aligned} f_x &= 3(x^2 - y) = 0 \\ f_y &= 3(-x + y^2) = 0 \end{aligned} \right\}$ Solving by substitution yields two critical points $(0, 0)$ and $(1, 1)$.
2. $f_{xx} = 6x$
3. $f_{yy} = 6y$
4. $f_{xy} = -3$
5. At the critical point $(0, 0)$, $f_{xx} f_{yy} - (f_{xy})^2 < 0$.
6. Therefore, $(0, 0, 0)$ is a saddle point.
7. At the critical point $(1, 1)$, $f_{xx} = 6 > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$.
8. Therefore, $(1, 1, -1)$ is a relative minimum.

Question 10

Examine the function for relative extrema. Use a computer algebra system to graph the function and confirm your results.

$$f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$$

1. $f_x = y - \frac{1}{x^2}$
2. $= 0$
3. $x^2y = 1$
4. $f_y = x - \frac{1}{y^2}$
5. $= 0$
6. $xy^2 = 1$
7. Thus, $x^2y = xy^2$ or
8. $x = y$
9. and substitution yields the critical point $(1, 1)$.
10. $f_{xx} = \frac{2}{x^3}$
11. $f_{xy} = 1$
12. $f_{yy} = \frac{2}{y^3}$
13. At the critical point $(1, 1)$, $f_{xx} = 2 > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = 3 > 0$.
14. Thus, $(1, 1, 3)$ is a relative minimum.
- 15.



Question 11

Find all critical points of the following function.

$$f(x,y) = \frac{1}{3}x^3 - 7y^3 - 5x + 21y - 7$$

An interior point (a,b) in the domain of f is a critical point of f if either $f_x(a,b) = f_y(a,b) = 0$, or one (or both) of f_x or f_y does not exist at (a,b) .

Find the partial derivatives f_x and f_y .

$$f_x = x^2 - 5$$

$$f_y = -21y^2 + 21$$

Set each derivative equal to zero and solve the system of equations for x and y .

$$x^2 - 5 = 0$$

$$-21y^2 + 21 = 0$$

Note that there are no y -terms in the equation for f_x , and no x -terms in the equation for f_y . Also note that there are two solutions for each equation. This means that there are four ordered pairs in total (every possible combination of each solution for x and y).

Solve the equation $f_x = 0$ for x .

$$x^2 - 5 = 0$$

$$x = \pm\sqrt{5}$$

Solve the equation $f_y = 0$ for y .

$$-21y^2 + 21 = 0$$

$$y = \pm 1$$

Since x can be $\pm\sqrt{5}$ and y can be ± 1 , there are four critical points for $f(x,y)$ at $(\sqrt{5}, 1)$, $(\sqrt{5}, -1)$, $(-\sqrt{5}, 1)$, and $(-\sqrt{5}, -1)$.

Question 12

Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4$$

Solution

Since

$$\begin{aligned} f_x(x, y) &= 4y - 4x^3 \\ f_y(x, y) &= 4x - 4y^3 \end{aligned} \tag{1}$$

the critical points of f have coordinates satisfying the equations

$$\begin{aligned} 4y - 4x^3 &= 0 & y &= x^3 \\ 4x - 4y^3 &= 0 & \text{or} & x = y^3 \end{aligned} \tag{2}$$

Substituting the top equation in the bottom yields $x = (x^3)^3$ or, equivalently, $x^9 - x = 0$ or $x(x^8 - 1) = 0$, which has solutions $x = 0, x = 1, x = -1$. Substituting these values in the top equation of (2), we obtain the corresponding y -values $y = 0, y = 1, y = -1$. Thus,

the critical points of f are $(0, 0), (1, 1),$ and $(-1, -1)$.

From (1),

$$f_{xx}(x, y) = -12x^2, \quad f_{yy}(x, y) = -12y^2, \quad f_{xy}(x, y) = 4$$

which yields the following table:

CRITICAL POINT (x_0, y_0)	$f_{xx}(x_0, y_0)$	$f_{yy}(x_0, y_0)$	$f_{xy}(x_0, y_0)$	$D = f_{xx}f_{yy} - f_{xy}^2$
$(0, 0)$	0	0	4	-16
$(1, 1)$	-12	-12	4	128
$(-1, -1)$	-12	-12	4	128

At the points $(1, 1)$ and $(-1, -1)$, we have $D > 0$ and $f_{xx} < 0$, so relative maxima occur at these critical points. At $(0, 0)$ there is a saddle point since $D < 0$.