MATH152 CALCULUS II TUTORIAL – VIII

(11.12.2015)

Question 1:

Use the gradient to find the directional derivative of the function at P in the direction of Q.

$f(x, y) = e^{-x} \cos y, \quad P(0, 0), \ Q(2, 1)$

1. $\overrightarrow{PQ} = 2\mathbf{i} + \mathbf{j}$
2. $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$
3. $\nabla f(x, y) = -e^{-x} \cos y\mathbf{i} - e^{-x} \sin y\mathbf{j}$
4. $\nabla f(0, 0) = -\mathbf{i}$
5. $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$
6. $\quad = -\frac{2}{\sqrt{5}}$
7. $\quad = -\frac{2\sqrt{5}}{5}$

Question 2 :

Find the directional derivative of the function at P in the direction of Q.

$h(x, y, z) = \ln(x + y + z), \quad P(1, 0, 0), \ Q(4, 3, 1)$

1. $\mathbf{v} = 3\mathbf{i} + 3\mathbf{j} + \mathbf{k}$
2. $\nabla h = \frac{1}{x + y + z}(\mathbf{i} + \mathbf{j} + \mathbf{k})$
3. At $(1, 0, 0)$, $\nabla h = \mathbf{i} + \mathbf{j} + \mathbf{k}$.
4. $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$
5. $\quad = \frac{1}{\sqrt{19}}(3\mathbf{i} + 3\mathbf{j} + \mathbf{k})$
6. $D_{\mathbf{u}} h = \nabla h \cdot \mathbf{u}$
7. $\quad = \frac{7}{\sqrt{19}}$
8. $\quad = \frac{7\sqrt{19}}{19}$

Question 3 :

Find the gradient of the function and the maximum value of the directional derivative at the given point.

Function	*Point*
$g(x, y) = \ln \sqrt[3]{x^2 + y^2}$	$(1, 2)$

1. $g(x, y) = \ln \sqrt[3]{x^2 + y^2}$
2. $\quad = \frac{1}{3}\ln(x^2 + y^2)$
3. $\nabla g(x, y) = \frac{1}{3}\left[\frac{2x}{x^2 + y^2}\mathbf{i} + \frac{2y}{x^2 + y^2}\mathbf{j}\right]$
4. $\nabla g(1, 2) = \frac{1}{3}\left(\frac{2}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right)$
5. $\quad = \frac{2}{15}(\mathbf{i} + 2\mathbf{j})$
6. $\|\nabla g(1, 2)\| = \frac{2\sqrt{5}}{15}$

Question 4:

Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

(a) increases most rapidly at the point (1, 1).

(b) decreases most rapidly at (1, 1).

(c) What are the directions of zero change in f at (1, 1)?

Solution

(a) The function increases most rapidly in the direction of ∇f at (1, 1). The gradient there is

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

(b) The function decreases most rapidly in the direction of $-\nabla f$ at (1, 1), which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

(c) The directions of zero change at (1, 1) are the directions orthogonal to ∇f:

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

See Figure 14.29.

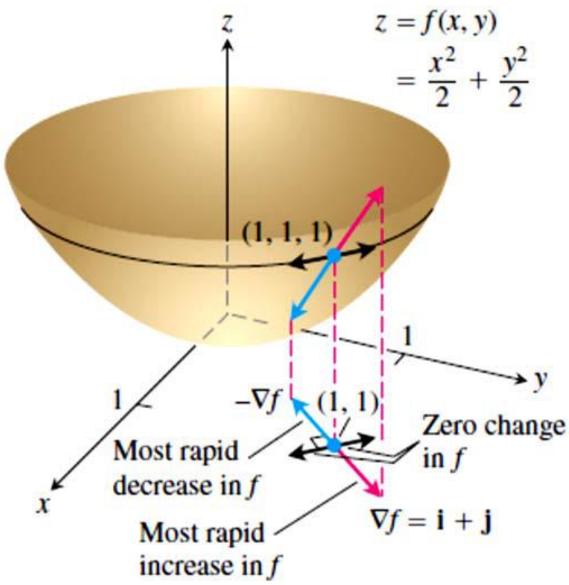


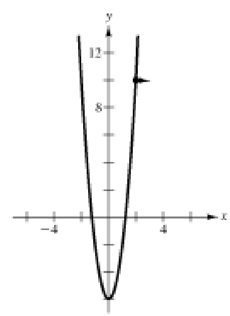
FIGURE 14.29 The direction in which $f(x, y)$ increases most rapidly at (1, 1) is the direction of $\nabla f|_{(1,1)} = \mathbf{i} + \mathbf{j}$. It corresponds to the direction of steepest ascent on the surface at (1, 1, 1) (Example 3).

<u>Question 5:</u>

Use the gradient to find a unit normal vector to the graph of the equation at the given point. Sketch your results.

$4x^2 - y = 6, \quad (2, 10)$

1. $f(x, y) = 4x^2 - y$
2. $\nabla f(x, y) = 8x\mathbf{i} - \mathbf{j}$
3. $\nabla f(2, 10) = 16\mathbf{i} - \mathbf{j}$
4. $\dfrac{\nabla f(2, 10)}{\|\nabla f(2, 10)\|} = \dfrac{1}{\sqrt{257}}(16\mathbf{i} - \mathbf{j})$
5. $\quad = \dfrac{\sqrt{257}}{257}(16\mathbf{i} - \mathbf{j})$
6.



<u>Question 6:</u>

Find a unit normal vector to the surface at the given point. [*Hint:* Normalize the gradient vector $\nabla F(x, y, z)$.]

Surface	*Point*
$z = \sqrt{x^2 + y^2}$	(3, 4, 5)

1. $F(x, y, z) = \sqrt{x^2 + y^2} - z$
2. $\nabla F(x, y, z) = \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{j} - \mathbf{k}$
3. $\nabla F(3, 4, 5) = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j} - \mathbf{k}$
4. $\mathbf{n} = \dfrac{\nabla F}{\|\nabla F\|}$
5. $\quad = \dfrac{5}{5\sqrt{2}}\left(\dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j} - \mathbf{k}\right)$
6. $\quad = \dfrac{1}{5\sqrt{2}}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$
7. $\quad = \dfrac{\sqrt{2}}{10}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$

Question 7:

Find an equation of the tangent plane to the surface at the given point.

$h(x, y) = \ln\sqrt{x^2 + y^2}$, $(3, 4, \ln 5)$

1. $H(x, y, z) = \ln\sqrt{x^2 + y^2} - z$
2. $= \frac{1}{2}\ln(x^2 + y^2) - z$
3. $H_x(x, y, z) = \frac{x}{x^2 + y^2}$
4. $H_y(x, y, z) = \frac{y}{x^2 + y^2}$
5. $H_z(x, y, z) = -1$
6. $H_x(3, 4, \ln 5) = \frac{3}{25}$
7. $H_y(3, 4, \ln 5) = \frac{4}{25}$

Question : 8

Find an equation of the tangent plane to the surface at the given point.

$xy^2 + 3x - z^2 = 4$, $(2, 1, -2)$

1. $F(x, y, z) = xy^2 + 3x - z^2 - 4$
2. $F_x(x, y, z) = y^2 + 3$
3. $F_y(x, y, z) = 2xy$
4. $F_z(x, y, z) = -2z$
5. $F_x(2, 1, -2) = 4$
6. $F_y(2, 1, -2) = 4$
7. $F_z(2, 1, -2) = 4$
8. $4(x - 2) + 4(y - 1) + 4(z + 2) = 0$
9. $x + y + z = 1$

Question 9

Find an equation of the tangent plane and find symmetric equations of the normal line to the surface at the given point.

$xy - z = 0$, $(-2, -3, 6)$

1. $F(x, y, z) = xy - z$
2. $F_x(x, y, z) = y$
3. $F_y(x, y, z) = x$
4. $F_z(x, y, z) = -1$
5. $F_x(-2, -3, 6) = -3$
6. $F_y(-2, -3, 6) = -2$
7. $F_z(-2, -3, 6) = -1$
8. Direction numbers: 3, 2, 1
9. Plane: $3(x + 2) + 2(y + 3) + (z - 6) = 0$
10. $3x + 2y + z = -6$
11. Line: $\frac{x + 2}{3} = \frac{y + 3}{2} = \frac{z - 6}{1}$

Question : 10

Find an equation of the tangent plane and find symmetric equations of the normal line to the surface at the given point.

$z = \arctan\frac{y}{x}$, $\left(1, 1, \frac{\pi}{4}\right)$

1. $F(x, y, z) = \arctan\frac{y}{x} - z$
2. $F_x(x, y, z) = \frac{-y}{x^2 + y^2}$
3. $F_y(x, y, z) = \frac{x}{x^2 + y^2}$
4. $F_z(x, y, z) = -1$
5. $F_x\left(1, 1, \frac{\pi}{4}\right) = -\frac{1}{2}$
6. $F_y\left(1, 1, \frac{\pi}{4}\right) = \frac{1}{2}$

Question : 11

Determine whether there is a relative maximum, a relative minimum, a saddle point, or insufficient information to determine the nature of the function $f(x, y)$ at the critical point (x_0, y_0).

$f_{xx}(x_0, y_0) = -9$, $f_{yy}(x_0, y_0) = 6$, $f_{xy}(x_0, y_0) = 10$

1. $f_{xx} f_{yy} - (f_{xy})^2 = (-9)(6) - 10^2$
2. < 0
3. f has a saddle point at (x_0, y_c).

Question : 12

Examine the function for relative extrema and saddle points.

$h(x, y) = x^2 - y^2 - 2x - 4y - 4$

1. $h_x = 2x - 2$
2. $= 2(x - 1)$
3. $= 0$ when $x = 1$.
4. $h_y = -2y - 4$
5. $= -2(y + 2)$
6. $= 0$ when $y = -2$.
7. $h_{xx} = 2$
8. $h_{yy} = -2$
9. $h_{xy} = 0$
10. At the critical point $(1, -2)$, $h_{xx} h_{yy} - (h_{xy})^2 < 0$.
11. Therefore, $(1, -2, -1)$ is a saddle point.

Question 13

Find the minimum distance from the point to the plane $2x + 3y + z = 12$. (*Hint:* To simplify the computations, minimize the square of the distance.)

$(0, 0, 0)$

1. A point on the plane is given by $(x, y, 12 - 2x - 3y)$.
2. The square of the distance from the origin to this point is

   $S = x^2 + y^2 + (12 - 2x - 3y)^2$
3. $S_x = 2x + 2(12 - 2x - 3y)(-2)$
4. $S_y = 2y + 2(12 - 2x - 3y)(-3)$.
5. From the equations $S_x = 0$ and $S_y = 0$, we obtain the system
6. $5x + 6y = 24$
7. $3x + 5y = 18$.
8. Solving simultaneously, we have $x = \frac{12}{7}$,
9. $y = \frac{18}{7}$,
10. $z = 12 - \frac{24}{7} - \frac{54}{7} = \frac{6}{7}$.
11. Therefore, the distance from the origin to $\left(\frac{12}{7}, \frac{18}{7}, \frac{6}{7}\right)$ is

$$\sqrt{\left(\frac{12}{7}\right)^2 + \left(\frac{18}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \frac{6\sqrt{14}}{7}.$$

Question 14

Find all critical points of the following function.

$$f(x,y) = \frac{1}{3}x^3 - 7y^3 - 5x + 21y - 7$$

An interior point (a,b) in the domain of f is a critical point of f if either $f_x(a,b) = f_y(a,b) = 0$, or one (or both) of f_x or f_y does not exist at (a,b).

Find the partial derivatives f_x and f_y.

$f_x = x^2 - 5$

$f_y = -21y^2 + 21$

Set each derivative equal to zero and solve the system of equations for x and y.

$$\begin{aligned} x^2 - 5 &= 0 \\ -21y^2 + 21 &= 0 \end{aligned}$$

Note that there are no y-terms in the equation for f_x, and no x-terms in the equation for f_y. Also note that there are two solutions for each equation. This means that there are four ordered pairs in total (every possible combination of each solution for x and y).

Solve the equation $f_x = 0$ for x.

$$\begin{aligned} x^2 - 5 &= 0 \\ x &= \pm\sqrt{5} \end{aligned}$$

Solve the equation $f_y = 0$ for y.

$$\begin{aligned} -21y^2 + 21 &= 0 \\ y &= \pm 1 \end{aligned}$$

Since x can be $\pm\sqrt{5}$ and y can be ± 1, there are four critical points for $f(x,y)$ at $(\sqrt{5},1)$, $(\sqrt{5},-1)$, $(-\sqrt{5},1)$, and $(-\sqrt{5},-1)$.

Question 15

Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4$$

Solution

Since

$$f_x(x, y) = 4y - 4x^3$$
$$f_y(x, y) = 4x - 4y^3 \qquad (1)$$

the critical points of f have coordinates satisfying the equations

$$4y - 4x^3 = 0 \quad \text{or} \quad y = x^3$$
$$4x - 4y^3 = 0 \qquad\quad x = y^3 \qquad (2)$$

Substituting the top equation in the bottom yields $x = (x^3)^3$ or, equivalently, $x^9 - x = 0$ or $x(x^8 - 1) = 0$, which has solutions $x = 0, x = 1, x = -1$. Substituting these values in the top equation of (2), we obtain the corresponding y-values $y = 0, y = 1, y = -1$. Thus,

the critical points of f are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

From (1),

$$f_{xx}(x, y) = -12x^2, \quad f_{yy}(x, y) = -12y^2, \quad f_{xy}(x, y) = 4$$

which yields the following table:

CRITICAL POINT (x_0, y_0)	$f_{xx}(x_0, y_0)$	$f_{yy}(x_0, y_0)$	$f_{xy}(x_0, y_0)$	$D = f_{xx}f_{yy} - f_{xy}^2$
(0, 0)	0	0	4	−16
(1, 1)	−12	−12	4	128
(−1, −1)	−12	−12	4	128

At the points $(1, 1)$ and $(-1, -1)$, we have $D > 0$ and $f_{xx} < 0$, so relative maxima occur at these critical points. At $(0, 0)$ there is a saddle point since $D < 0$.