

## MATH152 CALCULUS II TUTORIAL – VIII

(26.04.2019)

### Question 1:

Use the gradient to find the directional derivative of the function at  $P$  in the direction of  $Q$ .

$$f(x, y) = e^{-x} \cos y, \quad P(0, 0), \quad Q(2, 1)$$

1.  $\overrightarrow{PQ} = 2\mathbf{i} + \mathbf{j}$
2.  $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$
3.  $\nabla f(x, y) = -e^{-x} \cos y \mathbf{i} - e^{-x} \sin y \mathbf{j}$
4.  $\nabla f(0, 0) = -\mathbf{i}$
5.  $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$
6.  $= -\frac{2}{\sqrt{5}}$
7.  $= -\frac{2\sqrt{5}}{5}$

### Question 2:

Find the directional derivative of the function at  $P$  in the direction of  $\mathbf{v}$ .

$$f(x, y, z) = xy + yz + xz, \quad P(1, 1, 1), \quad \mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

1.  $\nabla f(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$
2.  $\nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
3.  $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$
4.  $= \frac{\sqrt{6}}{3}\mathbf{i} + \frac{\sqrt{6}}{6}\mathbf{j} - \frac{\sqrt{6}}{6}\mathbf{k}$
5.  $D_{\mathbf{u}}f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \mathbf{u}$
6.  $= \frac{2\sqrt{6}}{3}$

### Question 3:

Find the gradient of the function and the maximum value of the directional derivative at the given point.

<i>Function</i>	<i>Point</i>
$g(x, y) = \ln \sqrt[3]{x^2 + y^2}$	$(1, 2)$

1.  $g(x, y) = \ln \sqrt[3]{x^2 + y^2}$
2.  $= \frac{1}{3} \ln(x^2 + y^2)$
3.  $\nabla g(x, y) = \frac{1}{3} \left[ \frac{2x}{x^2 + y^2} \mathbf{i} + \frac{2y}{x^2 + y^2} \mathbf{j} \right]$
4.  $\nabla g(1, 2) = \frac{1}{3} \left( \frac{2}{5} \mathbf{i} + \frac{4}{5} \mathbf{j} \right)$
5.  $= \frac{2}{15} (\mathbf{i} + 2\mathbf{j})$
6.  $\|\nabla g(1, 2)\| = \frac{2\sqrt{5}}{15}$

### Question 4:

Find the directions in which  $f(x, y) = (x^2/2) + (y^2/2)$

- (a) increases most rapidly at the point  $(1, 1)$ .
- (b) decreases most rapidly at  $(1, 1)$ .
- (c) What are the directions of zero change in  $f$  at  $(1, 1)$ ?

#### **Solution**

- (a) The function increases most rapidly in the direction of  $\nabla f$  at  $(1, 1)$ . The gradient there is

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{\|\mathbf{i} + \mathbf{j}\|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

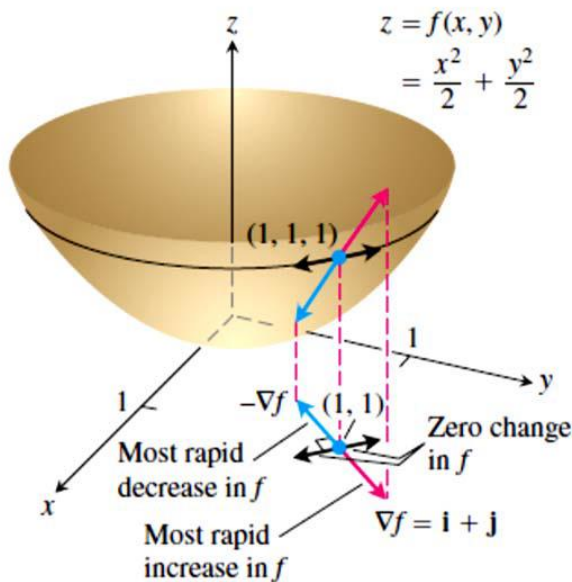
- (b) The function decreases most rapidly in the direction of  $-\nabla f$  at  $(1, 1)$ , which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

(c) The directions of zero change at  $(1, 1)$  are the directions orthogonal to  $\nabla f$ :

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

See Figure 14.29.



**FIGURE 14.29** The direction in which  $f(x, y)$  increases most rapidly at  $(1, 1)$  is the direction of  $\nabla f|_{(1,1)} = \mathbf{i} + \mathbf{j}$ . It corresponds to the direction of steepest ascent on the surface at  $(1, 1, 1)$  (Example 3).

### Question 5:

Find a unit normal vector to the surface at the given point. [Hint: Normalize the gradient vector  $\nabla F(x, y, z)$ .]

Surface	Point
$z = \sqrt{x^2 + y^2}$	$(3, 4, 5)$

- $F(x, y, z) = \sqrt{x^2 + y^2} - z$
- $\nabla F(x, y, z) = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j} - \mathbf{k}$
- $\nabla F(3, 4, 5) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}$
- $\mathbf{n} = \frac{\nabla F}{\|\nabla F\|}$
- $= \frac{5}{5\sqrt{2}}\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}\right)$
- $= \frac{1}{5\sqrt{2}}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$
- $= \frac{\sqrt{2}}{10}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$

### Question : 6

Find an equation of the tangent plane to the surface at the given point.

$$x^2 + 4y^2 + z^2 = 36, \quad (2, -2, 4)$$

- $F(x, y, z) = x^2 + 4y^2 + z^2 - 36$
- $F_x(x, y, z) = 2x$
- $F_y(x, y, z) = 8y$
- $F_z(x, y, z) = 2z$
- $F_x(2, -2, 4) = 4$
- $F_y(2, -2, 4) = -16$
- $F_z(2, -2, 4) = 8$
- $4(x - 2) - 16(y + 2) + 8(z - 4) = 0$
- $(x - 2) - 4(y + 2) + 2(z - 4) = 0$
- $x - 4y + 2z = 18$

### Question 7

Find an equation of the tangent plane and find symmetric equations of the normal line to the surface at the given point.

$$x^2 + y^2 + z = 9, \quad (1, 2, 4)$$

1.  $F(x, y, z) = x^2 + y^2 + z - 9$
2.  $F_x(x, y, z) = 2x$
3.  $F_y(x, y, z) = 2y$
4.  $F_z(x, y, z) = 1$
5.  $F_x(1, 2, 4) = 2$
6.  $F_y(1, 2, 4) = 4$
7.  $F_z(1, 2, 4) = 1$
8. Direction numbers: 2, 4, 1
9. Plane:  $2(x - 1) + 4(y - 2) + (z - 4) = 0$
10.  $2x + 4y + z = 14$
11. Line:  $\frac{x - 1}{2} = \frac{y - 2}{4} = \frac{z - 4}{1}$

### Question : 9

Examine the function for relative extrema and saddle points.

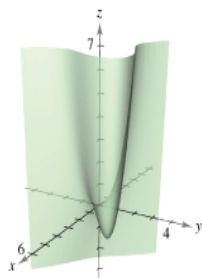
$$h(x, y) = x^2 - y^2 - 2x - 4y - 4$$

1.  $h_x = 2x - 2$
2.  $= 2(x - 1)$
3.  $= 0$  when  $x = 1$ .
4.  $h_y = -2y - 4$
5.  $= -2(y + 2)$
6.  $= 0$  when  $y = -2$ .
7.  $h_{xx} = 2$
8.  $h_{yy} = -2$
9.  $h_{xy} = 0$
10. At the critical point  $(1, -2)$ ,  $h_{xx} h_{yy} - (h_{xy})^2 < 0$ .
11. Therefore,  $(1, -2, -1)$  is a saddle point.

### Question : 8

Examine the function for relative extrema and saddle points.

$$f(x, y) = x^3 - 3xy + y^3$$



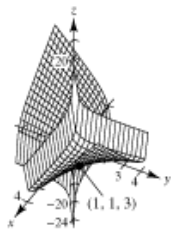
1.  $\left. \begin{aligned} f_x &= 3(x^2 - y) = 0 \\ f_y &= 3(-x + y^2) = 0 \end{aligned} \right\}$  Solving by substitution yields two critical points  $(0, 0)$  and  $(1, 1)$ .
2.  $f_{xx} = 6x$
3.  $f_{yy} = 6y$
4.  $f_{xy} = -3$
5. At the critical point  $(0, 0)$ ,  $f_{xx} f_{yy} - (f_{xy})^2 < 0$ .
6. Therefore,  $(0, 0, 0)$  is a saddle point.
7. At the critical point  $(1, 1)$ ,  $f_{xx} = 6 > 0$  and  $f_{xx} f_{yy} - (f_{xy})^2 > 0$ .
8. Therefore,  $(1, 1, -1)$  is a relative minimum.

### Question 10

Examine the function for relative extrema. Use a computer algebra system to graph the function and confirm your results.

$$f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$$

1.  $f_x = y - \frac{1}{x^2}$
2.  $= 0$
3.  $x^2y = 1$
4.  $f_y = x - \frac{1}{y^2}$
5.  $= 0$
6.  $xy^2 = 1$
7. Thus,  $x^2y = xy^2$  or
8.  $x = y$
9. and substitution yields the critical point  $(1, 1)$ .
10.  $f_{xx} = \frac{2}{x^3}$
11.  $f_{xy} = 1$
12.  $f_{yy} = \frac{2}{y^3}$
13. At the critical point  $(1, 1)$ ,  $f_{xx} = 2 > 0$  and  $f_{xx}f_{yy} - (f_{xy})^2 = 3 > 0$ .
14. Thus,  $(1, 1, 3)$  is a relative minimum.
- 15.



### Question 11

Find all critical points of the following function.

$$f(x,y) = \frac{1}{3}x^3 - 7y^3 - 5x + 21y - 7$$

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An interior point  $(a,b)$  in the domain of  $f$  is a critical point of  $f$  if either  $f_x(a,b) = f_y(a,b) = 0$ , or one (or both) of  $f_x$  or  $f_y$  does not exist at  $(a,b)$ .

Find the partial derivatives  $f_x$  and  $f_y$ .

$$f_x = x^2 - 5$$

$$f_y = -21y^2 + 21$$

Set each derivative equal to zero and solve the system of equations for  $x$  and  $y$ .

$$x^2 - 5 = 0$$

$$-21y^2 + 21 = 0$$

Note that there are no  $y$ -terms in the equation for  $f_x$ , and no  $x$ -terms in the equation for  $f_y$ . Also note that there are two solutions for each equation. This means that there are four ordered pairs in total (every possible combination of each solution for  $x$  and  $y$ ).

Solve the equation  $f_x = 0$  for  $x$ .

$$x^2 - 5 = 0$$

$$x = \pm\sqrt{5}$$

Solve the equation  $f_y = 0$  for  $y$ .

$$-21y^2 + 21 = 0$$

$$y = \pm 1$$

Since  $x$  can be  $\pm\sqrt{5}$  and  $y$  can be  $\pm 1$ , there are four critical points for  $f(x,y)$  at  $(\sqrt{5}, 1)$ ,  $(\sqrt{5}, -1)$ ,  $(-\sqrt{5}, 1)$ , and  $(-\sqrt{5}, -1)$ .

## **Question 12**

Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4$$

### Solution

Since

$$\begin{aligned} f_x(x, y) &= 4y - 4x^3 \\ f_y(x, y) &= 4x - 4y^3 \end{aligned} \tag{1}$$

the critical points of  $f$  have coordinates satisfying the equations

$$\begin{aligned} 4y - 4x^3 &= 0 & y &= x^3 \\ 4x - 4y^3 &= 0 & \text{or} & x = y^3 \end{aligned} \tag{2}$$

Substituting the top equation in the bottom yields  $x = (x^3)^3$  or, equivalently,  $x^9 - x = 0$  or  $x(x^8 - 1) = 0$ , which has solutions  $x = 0$ ,  $x = 1$ ,  $x = -1$ . Substituting these values in the top equation of (2), we obtain the corresponding  $y$ -values  $y = 0$ ,  $y = 1$ ,  $y = -1$ . Thus,

the critical points of  $f$  are  $(0, 0)$ ,  $(1, 1)$ , and  $(-1, -1)$ .

From (1),

$$f_{xx}(x, y) = -12x^2, \quad f_{yy}(x, y) = -12y^2, \quad f_{xy}(x, y) = 4$$

which yields the following table:

CRITICAL POINT $(x_0, y_0)$	$f_{xx}(x_0, y_0)$	$f_{yy}(x_0, y_0)$	$f_{xy}(x_0, y_0)$	$D = f_{xx}f_{yy} - f_{xy}^2$
$(0, 0)$	0	0	4	-16
$(1, 1)$	-12	-12	4	128
$(-1, -1)$	-12	-12	4	128

At the points  $(1, 1)$  and  $(-1, -1)$ , we have  $D > 0$  and  $f_{xx} < 0$ , so relative maxima occur at these critical points. At  $(0, 0)$  there is a saddle point since  $D < 0$ .