

## MATH152 CALCULUS II TUTORIAL – VII

(08.04.2016)

### Question 1:

Describe the domain and range of the function.

$$f(x, y) = \sqrt{4 - x^2 - y^2}$$

1. Domain:  $4 - x^2 - y^2 \geq 0$
2.  $x^2 + y^2 \leq 4$
3.  $\{(x, y): x^2 + y^2 \leq 4\}$
4. Range:  $0 \leq z \leq 2$

### Question 2:

Find the limit and discuss the continuity of the function (if it exists).

$$\lim_{(x, y) \rightarrow (1, 1)} \frac{xy}{x^2 + y^2}$$

1.  $\lim_{(x, y) \rightarrow (1, 1)} \frac{xy}{x^2 + y^2} = \frac{1}{2}$
2. Continuous except at  $(0, 0)$ .

### Question 3:

Find the limit and discuss the continuity of the function (if it exists).

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{-4x^2y}{x^4 + y^2}$$

1. For  $y = x^2$ ,  $\frac{-4x^2y}{x^4 + y^2} = \frac{-4x^4}{x^4 + x^4}$
2.  $= -2$ , for  $x \neq 0$
3. For  $y = 0$ ,  $\frac{-4x^2y}{x^4 + y^2} = 0$ , for  $x \neq 0$
4. Thus, the limit does not exist.
5. Continuous except at  $(0, 0)$ .

### Question 4:

Investigate the limit  $\lim_{(x, y) \rightarrow (0, 0)} \frac{(x + y)^2}{x^2 + y^2}$ .

### Solution:

The domain of the function is  $\{(x, y): (x, y) \neq (0, 0)\}$ ; therefore, the limit is at a boundary point outside the domain. Suppose we let  $(x, y)$  approach  $(0, 0)$  along the line  $y = mx$  for a fixed constant  $m$ . Substituting  $y = mx$  and noting that  $y \rightarrow 0$  as  $x \rightarrow 0$ , we have

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{(x + y)^2}{(x^2 + y^2)} = \lim_{x \rightarrow 0} \frac{(x + mx)^2}{(x^2 + m^2x^2)} = \lim_{x \rightarrow 0} \frac{x^2(1 + m)^2}{x^2(1 + m^2)} = \frac{(1 + m)^2}{(1 + m^2)}$$

The constant  $m$  determines the direction of approach to  $(0, 0)$ . Therefore, depending on  $m$ , the function may approach any value in the interval  $[0, 2]$  (which is the range of  $(1 + m)^2 / (1 + m^2)$ ) as  $(x, y)$  approaches  $(0, 0)$  (Figure 12.44). For example, if  $m = 0$ , the corresponding limit is 1 and if  $m = -1$ , the limit is 0. Because the function approaches different values along different paths, we conclude that the *limit does not exist*.

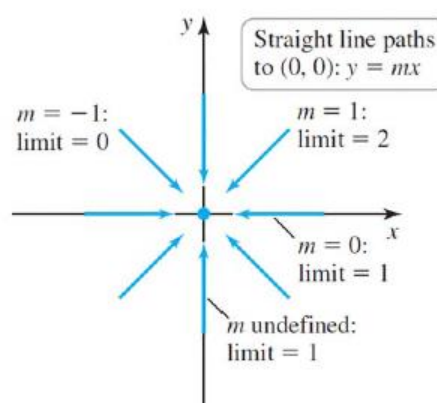


FIGURE 12.44

### Question 5:

Discuss the continuity of the functions  $f$  and  $g$ . Explain any differences.

$$f(x, y) = \begin{cases} \frac{x^2 + 2xy^2 + y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$$g(x, y) = \begin{cases} \frac{x^2 + 2xy^2 + y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 1, & (x, y) = (0, 0) \end{cases}$$

1.  $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \left( \frac{x^2 + 2xy^2 + y^2}{x^2 + y^2} \right)$

2.  $= \lim_{(x, y) \rightarrow (0, 0)} \left( 1 + \frac{2xy^2}{x^2 + y^2} \right)$

3.  $= 1$

4. (same limit for  $g$ )

5. Thus,  $f$  is not continuous at  $(0, 0)$ , whereas  $g$  is continuous at  $(0, 0)$ .

### Question 6:

Find both first partial derivatives.

$$z = \frac{x^2}{2y} + \frac{4y^2}{x}$$

1.  $\frac{\partial z}{\partial x} = \frac{2x}{2y} - \frac{4y^2}{x^2}$

2.  $= \frac{x^3 - 4y^3}{x^2y}$

3.  $\frac{\partial z}{\partial y} = -\frac{x^2}{2y^2} + \frac{8y}{x}$

4.  $= \frac{-x^3 + 16y^3}{2xy^2}$

### Question 7:

Find all first partial derivatives.

$$f(x, y) = e^x \cos y$$

1.  $f_x = e^x \cos y$

2.  $f_y = -e^x \sin y$

### Question 7:

Find all second partial derivatives and verify that the second mixed partials are equal.

$$h(x, y) = x \sin y + y \cos x$$

1.  $h_x = \sin y - y \sin x$

2.  $h_y = x \cos y + \cos x$

3.  $h_{xx} = -y \cos x$

4.  $h_{yy} = -x \sin y$

5.  $h_{xy} = \cos y - \sin x$

6.  $h_{yx} = \cos y - \sin x$

### Question : 8

Find  $dw/dt$  (a) using the appropriate Chain Rule and (b) by converting  $w$  to a function of  $t$  before differentiating.

$$w = xy + xz + yz, \quad x = t - 1, \quad y = t^2 - 1, \quad z = t$$

1. (a)  $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$

2.  $= (y + z) + (x + z)(2t) + (x + y)$

3.  $= (t^2 - 1 + t) + (t - 1 + t)(2t) + (t - 1 + t^2 - 1)$

4.  $= 3(2t^2 - 1)$

5. (b)  $w = (t - 1)(t^2 - 1) + (t - 1)t + (t^2 - 1)t$

6.  $\frac{dw}{dt} = 2t(t - 1) + (t^2 - 1) + 2t - 1 + 3t^2 - 1$

7.  $= 3(2t^2 - 1)$

**Question 9**

Find  $\partial w/\partial s$  and  $\partial w/\partial t$  when  $s = 1$  and  $t = 2\pi$  for the function given by

$$w = xy + yz + xz$$

where  $x = s \cos t$ ,  $y = s \sin t$ , and  $z = t$ .

**Solution**

$$\begin{aligned} \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s} \\ &= (y + z)(\cos t) + (x + z)(\sin t) + (y + x)(0) \\ &= (y + z)(\cos t) + (x + z)(\sin t). \end{aligned}$$

When  $s = 1$  and  $t = 2\pi$ , you have  $x = 1$ ,  $y = 0$ , and  $z = 2$   
 $\partial w/\partial s = (0 + 2\pi)(1) + (1 + 2\pi)(0) = 2\pi$ . Furthermore,

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} \\ &= (y + z)(-s \sin t) + (x + z)(s \cos t) + (y + x)(1) \end{aligned}$$

and for  $s = 1$  and  $t = 2\pi$ , it follows that

$$\begin{aligned} \frac{\partial w}{\partial t} &= (0 + 2\pi)(0) + (1 + 2\pi)(1) + (0 + 1)(1) \\ &= 2 + 2\pi. \end{aligned}$$

**Question : 10**

$$w = e^{xyz}, \quad x = 3u + v, \quad y = 3u - v, \quad z = u^2v$$

Use appropriate forms of the chain rule to find  $\partial w/\partial u$  and  $\partial w/\partial v$ .

**Solution**

$$\frac{\partial w}{\partial u} = yze^{xyz}(3) + xze^{xyz}(3) + xye^{xyz}(2uv) = e^{xyz}(3yz + 3xz + 2xyuv)$$

$$\frac{\partial w}{\partial v} = yze^{xyz}(1) + xze^{xyz}(-1) + xye^{xyz}(u^2) = e^{xyz}(yz - xz + xyu^2)$$

**Question 11**

Differentiate implicitly to find the first partial derivatives of  $z$ .

$$x^2 + 2yz + z^2 = 1$$

$$1. F(x, y, z) = x^2 + 2yz + z^2 - 1$$

$$2. = 0$$

$$3. \frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}$$

$$4. = \frac{-2x}{2y + 2z}$$

$$5. = \frac{-x}{y + z}$$

$$6. \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$$

$$7. = \frac{-2z}{2y + 2z}$$

$$8. = \frac{-z}{y + z}$$

**Question 12**

Differentiate implicitly to find  $dy/dx$ .

$$x^2 - 3xy + y^2 - 2x + y - 5 = 0$$

$$1. \frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$$

$$2. = -\frac{2x - 3y - 2}{-3x + 2y + 1}$$

$$3. = \frac{3y - 2x + 2}{2y - 3x + 1}$$

### Question 13

For  $f(x, y)$ , find all values of  $x$  and  $y$  such that  $f_x(x, y) = 0$  and  $f_y(x, y) = 0$  simultaneously.

$$f(x, y) = x^2 + 4xy + y^2 - 4x + 16y + 3$$

1.  $f_x(x, y) = 2x + 4y - 4$
2.  $f_y(x, y) = 4x + 2y + 16$
3.  $f_x = f_y = 0$ :
4.  $2x + 4y = 4$
5.  $4x + 2y = -16$
6. Solving for  $x$  and  $y$ ,  $x = -6$  and
7.  $y = 4$ .

### Question 14

Find the four second partial derivatives of

$$f(x, y) = 3x^4y - 2xy + 5xy^3.$$

**Solution**

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(3x^4y - 2xy + 5xy^3) = 12x^3y - 2y + 5y^3$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(3x^4y - 2xy + 5xy^3) = 3x^4 - 2x + 15xy^2.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(12x^3y - 2y + 5y^3) = 36x^2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^4 - 2x + 15xy^2) = 30xy$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^4 - 2x + 15xy^2) = 12x^3 - 2 + 15y^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(12x^3y - 2y + 5y^3) = 12x^3 - 2 + 15y^2$$

### Question 15

Show that  $f_{xz} = f_{zx}$  and  $f_{xzz} = f_{zxx} = f_{zzx}$  for the function given by

$$f(x, y, z) = ye^x + x \ln z.$$

**Solution**

First partials:

$$f_x(x, y, z) = ye^x + \ln z, \quad f_z(x, y, z) = \frac{x}{z}$$

Second partials (note that the first two are equal):

$$f_{xz}(x, y, z) = \frac{1}{z}, \quad f_{zx}(x, y, z) = \frac{1}{z}, \quad f_{zz}(x, y, z) = -\frac{x}{z^2}$$

Third partials (note that all three are equal):

$$f_{xzz}(x, y, z) = -\frac{1}{z^2}, \quad f_{zxc}(x, y, z) = -\frac{1}{z^2}, \quad f_{zzx}(x, y, z) = -\frac{1}{z^2}$$

### Question 16

Find  $f_{yxyz}$  if  $f(x, y, z) = 1 - 2xy^2z + x^2y$ .

**Solution**

$$f_y = -4xyz + x^2$$

$$f_{yx} = -4yz + 2x$$

$$f_{yxy} = -4z$$

$$f_{yxyz} = -4$$