MATH152 CALCULUS II TUTORIAL – VII

(5.04.2019)

Question 1:

Describe the domain and range of the function.

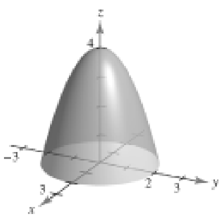
$f(x, y) = \ln(4 - x - y)$

1. Domain: $4 - x - y > 0$
2. $x + y < 4$
3. $\{(x, y): y < -x + 4\}$
4. Range: all real numbers

Question 2 :

Sketch the surface given by the function.

$z = 4 - x^2 - y^2$

1. Paraboloid
2. Domain: entire xy-plane
3. Range: $z \leq 4$
4. 

Question 3 :

Find the limit (if it exists). If the limit does not exist, explain why.

$$\lim_{(x, y)\to(0, 0)} \frac{x + y}{x^2 + y}$$

1. The limit does not exist because along the line $y = 0$ you have
2. $\displaystyle \lim_{(x, y)\to(0, 0)} \frac{x + y}{x^2 + y} = \lim_{(x, 0)\to(0, 0)} \frac{x}{x^2}$
3. $\displaystyle = \lim_{(x, 0)\to(0, 0)} \frac{1}{x}$
4. which does not exist.

Question 4:

Find the limit and discuss the continuity of the function.

$$\lim_{(x, y)\to(2, 4)} \frac{x + y}{x - y}$$

1. $\displaystyle \lim_{(x, y)\to(2, 4)} \frac{x + y}{x - y} = \frac{2 + 4}{2 - 4}$
2. $= -3$
3. Continuous for $x \neq y$

Question 5:

Investigate the limit $\displaystyle \lim_{(x,y)\to(0,0)} \frac{(x + y)^2}{x^2 + y^2}$.

Solution:

The domain of the function is $\{(x, y): (x, y) \neq (0, 0)\}$; therefore, the limit is at a boundary point outside the domain. Suppose we let (x, y) approach $(0, 0)$ along the line $y = mx$ for a fixed constant m. Substituting $y = mx$ and noting that $y \to 0$ as $x \to 0$, we have

$$\lim_{(x,y)\to(0,0)} \frac{(x + y)^2}{(x^2 + y^2)} = \lim_{x\to 0} \frac{(x + mx)^2}{(x^2 + m^2x^2)} = \lim_{x\to 0} \frac{x^2(1 + m)^2}{x^2(1 + m^2)} = \frac{(1 + m)^2}{(1 + m^2)}.$$

The constant m determines the direction of approach to $(0, 0)$. Therefore, depending on m, the function may approach any value in the interval $[0, 2]$ (which is the range of $(1 + m)^2/(1 + m^2)$) as (x, y) approaches $(0, 0)$ (Figure 12.44). For example, if $m = 0$, the corresponding limit is 1 and if $m = -1$, the limit is 0. Because the function approaches different values along different paths, we conclude that the *limit does not exist*.

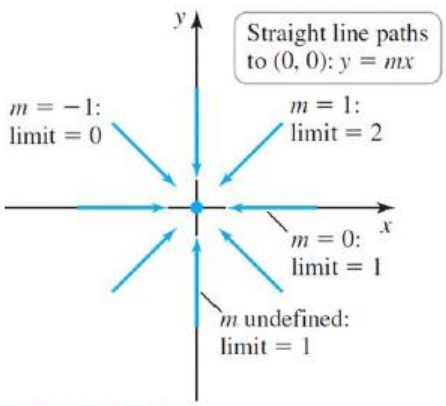


FIGURE 12.44

Question 6:

Discuss the continuity of the functions f and g. Explain any differences.

$$f(x, y) = \begin{cases} \dfrac{x^2 + 2xy^2 + y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$$g(x, y) = \begin{cases} \dfrac{x^2 + 2xy^2 + y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 1, & (x, y) = (0, 0) \end{cases}$$

1. $\displaystyle \lim_{(x, y)\to(0, 0)} f(x, y) = \lim_{(x, y)\to(0, 0)} \left(\frac{x^2 + 2xy^2 + y^2}{x^2 + y^2}\right)$
2. $\displaystyle \qquad = \lim_{(x, y)\to(0, 0)} \left(1 + \frac{2xy^2}{x^2 + y^2}\right)$
3. $\qquad = 1$
4. (same limit for g)
5. Thus, f is not continuous at $(0, 0)$, whereas g is continuous at $(0, 0)$.

Question 7:

Find both first partial derivatives.

$z = \ln(x^2 + y^2)$

1. $\dfrac{\partial z}{\partial x} = \dfrac{2x}{x^2 + y^2}$
2. $\dfrac{\partial z}{\partial y} = \dfrac{2y}{x^2 + y^2}$

Question 8:

Evaluate f_x and f_y at the given point.

$f(x, y) = \dfrac{xy}{x - y}, \quad (2, -2)$

1. $f_x(x, y) = \dfrac{y(x - y) - xy}{(x - y)^2}$
2. $\qquad = \dfrac{-y^2}{(x - y)^2}$
3. At $(2, -2)$: $f_x(2, -2) = -\dfrac{1}{4}$
4. $f_y(x, y) = \dfrac{x(x - y) + xy}{(x - y)^2}$
5. $\qquad = \dfrac{x^2}{(x - y)^2}$
6. At $(2, -2)$: $f_y(2, -2) = \dfrac{1}{4}$

Question : 9

Find dw/dt (a) using the appropriate Chain Rule and (b) by converting w to a function of t before differentiating.

$w = xy + xz + yz, \quad x = t - 1, \quad y = t^2 - 1, \quad z = t$

1. (a) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt}$
2. $\qquad = (y + z) + (x + z)(2t) + (x + y)$
3. $\qquad = (t^2 - 1 + t) + (t - 1 + t)(2t) + (t - 1 + t^2 - 1)$
4. $\qquad = 3(2t^2 - 1)$
5. (b) $\quad w = (t - 1)(t^2 - 1) + (t - 1)t + (t^2 - 1)t$
6. $\quad \dfrac{dw}{dt} = 2t(t - 1) + (t^2 - 1) + 2t - 1 + 3t^2 - 1$
7. $\qquad = 3(2t^2 - 1)$

Question 9

Find $\partial w/\partial s$ and $\partial w/\partial t$ when $s = 1$ and $t = 2\pi$ for the function given by

$$w = xy + yz + xz$$

where $x = s\cos t$, $y = s\sin t$, and $z = t$.

Solution

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= (y + z)(\cos t) + (x + z)(\sin t) + (y + x)(0)$$

$$= (y + z)(\cos t) + (x + z)(\sin t).$$

When $s = 1$ and $t = 2\pi$, you have $x = 1$, $y = 0$, and $z = 2$ $\partial w/\partial s = (0 + 2\pi)(1) + (1 + 2\pi)(0) = 2\pi$. Furthermore,

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$= (y + z)(-s\sin t) + (x + z)(s\cos t) + (y + x)(1)$$

and for $s = 1$ and $t = 2\pi$, it follows that

$$\frac{\partial w}{\partial t} = (0 + 2\pi)(0) + (1 + 2\pi)(1) + (0 + 1)(1)$$

$$= 2 + 2\pi.$$

Question : 10

$$w = e^{xyz}, \quad x = 3u + v, \quad y = 3u - v, \quad z = u^2 v$$

Use appropriate forms of the chain rule to find $\partial w/\partial u$ and $\partial w/\partial v$.

Solution

$$\frac{\partial w}{\partial u} = yze^{xyz}(3) + xze^{xyz}(3) + xye^{xyz}(2uv) = e^{xyz}(3yz + 3xz + 2xyu$$

$$\frac{\partial w}{\partial v} = yze^{xyz}(1) + xze^{xyz}(-1) + xye^{xyz}(u^2) = e^{xyz}(yz - xz + xyu^2)$$

Question 11

Differentiate implicitly to find the first partial derivatives of z.

$$x^2 + 2yz + z^2 = 1$$

1. $F(x, y, z) = x^2 + 2yz + z^2 - 1$
2. $= 0$
3. $\dfrac{\partial z}{\partial x} = -\dfrac{F_x(x, y, z)}{F_z(x, y, z)}$
4. $= \dfrac{-2x}{2y + 2z}$
5. $= \dfrac{-x}{y + z}$
6. $\dfrac{\partial z}{\partial y} = -\dfrac{F_y(x, y, z)}{F_z(x, y, z)}$
7. $= \dfrac{-2z}{2y + 2z}$
8. $= \dfrac{-z}{y + z}$

Question 12

Find the four second partial derivatives. Observe that the second mixed partials are equal.

$$z = \sqrt{x^2 + y^2}$$

1. $\dfrac{\partial z}{\partial x} = \dfrac{x}{\sqrt{x^2 + y^2}}$
2. $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{y^2}{(x^2 + y^2)^{3/2}}$
3. $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{-xy}{(x^2 + y^2)^{3/2}}$
4. $\dfrac{\partial z}{\partial y} = \dfrac{y}{\sqrt{x^2 + y^2}}$
5. $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{x^2}{(x^2 + y^2)^{3/2}}$
6. $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{-xy}{(x^2 + y^2)^{3/2}}$

Question 13

Differentiate implicitly to find dy/dx.

$x^2 - 3xy + y^2 - 2x + y - 5 = 0$

1. $\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$
2. $= -\frac{2x - 3y - 2}{-3x + 2y + 1}$
3. $= \frac{3y - 2x + 2}{2y - 3x + 1}$

Question 14

For $f(x, y)$, find all values of x and y such that $f_x(x, y) = 0$ and $f_y(x, y) = 0$ simultaneously.

$f(x, y) = x^2 + 4xy + y^2 - 4x + 16y + 3$

1. $f_x(x, y) = 2x + 4y - 4$
2. $f_y(x, y) = 4x + 2y + 16$
3. $f_x = f_y = 0$:
4. $2x + 4y = 4$
5. $4x + 2y = -16$
6. Solving for x and y, $x = -6$ and
7. $y = 4$.

Question 15

Find the four second partial derivatives of $f(x, y) = 3x^4y - 2xy + 5xy^3$.

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(3x^4y - 2xy + 5xy^3) = 12x^3y - 2y + 5y^3$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(3x^4y - 2xy + 5xy^3) = 3x^4 - 2x + 15xy^2.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(12x^3y - 2y + 5y^3) = 36x^2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^4 - 2x + 15xy^2) = 30xy$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^4 - 2x + 15xy^2) = 12x^3 - 2 + 15y^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(12x^3y - 2y + 5y^3) = 12x^3 - 2 + 15y^2.$$

Question 16

Show that $f_{xz} = f_{zx}$ and $f_{xzz} = f_{zxz} = f_{zzx}$ for the function given by

$f(x, y, z) = ye^x + x \ln z.$

Solution

First partials:

$$f_x(x, y, z) = ye^x + \ln z, \qquad f_z(x, y, z) = \frac{x}{z}$$

Second partials (note that the first two are equal):

$$f_{xz}(x, y, z) = \frac{1}{z}, \qquad f_{zx}(x, y, z) = \frac{1}{z}, \qquad f_{zz}(x, y, z) = -\frac{x}{z^2}$$

Third partials (note that all three are equal):

$$f_{xzz}(x, y, z) = -\frac{1}{z^2}, \qquad f_{zxz}(x, y, z) = -\frac{1}{z^2}, \qquad f_{zzx}(x, y, z) = -\frac{1}{z^2}$$

Question 17

Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Solution

$$f_y = -4xyz + x^2$$
$$f_{yx} = -4yz + 2x$$
$$f_{yxy} = -4z$$
$$f_{yxyz} = -4$$