

MATH152 CALCULUS II TUTORIAL – VII

(07.04.2017)

Question 1:

Describe the domain and range of the function.

$$f(x, y) = \ln(4 - x - y)$$

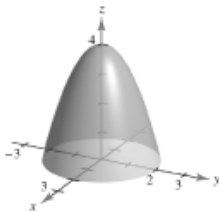
1. Domain: $4 - x - y > 0$
2. $x + y < 4$
3. $\{(x, y): y < -x + 4\}$
4. Range: all real numbers

Question 2 :

Sketch the surface given by the function.

$$z = 4 - x^2 - y^2$$

1. Paraboloid
2. Domain: entire xy -plane
3. Range: $z \leq 4$
- 4.



Question 3 :

Find the limit (if it exists). If the limit does not exist, explain why.

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{x + y}{x^2 + y}$$

1. The limit does not exist because along the line $y = 0$ you have
2. $\lim_{(x, y) \rightarrow (0, 0)} \frac{x + y}{x^2 + y} = \lim_{(x, 0) \rightarrow (0, 0)} \frac{x}{x^2}$
3. $= \lim_{(x, 0) \rightarrow (0, 0)} \frac{1}{x}$
4. which does not exist.

Question 4:

Find the limit and discuss the continuity of the function.

$$\lim_{(x, y) \rightarrow (2, 4)} \frac{x + y}{x - y}$$

1. $\lim_{(x, y) \rightarrow (2, 4)} \frac{x + y}{x - y} = \frac{2 + 4}{2 - 4}$
2. $= -3$
3. Continuous for $x \neq y$

Question 5:

Investigate the limit $\lim_{(x, y) \rightarrow (0, 0)} \frac{(x + y)^2}{x^2 + y^2}$.

Solution:

The domain of the function is $\{(x, y): (x, y) \neq (0, 0)\}$; therefore, the limit is at a boundary point outside the domain. Suppose we let (x, y) approach $(0, 0)$ along the line $y = mx$ for a fixed constant m . Substituting $y = mx$ and noting that $y \rightarrow 0$ as $x \rightarrow 0$, we have

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{(x + y)^2}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{(x + mx)^2}{x^2 + m^2 x^2} = \lim_{x \rightarrow 0} \frac{x^2(1 + m)^2}{x^2(1 + m^2)} = \frac{(1 + m)^2}{(1 + m^2)}$$

The constant m determines the direction of approach to $(0, 0)$. Therefore, depending on m , the function may approach any value in the interval $[0, 2]$ (which is the range of $(1 + m)^2 / (1 + m^2)$) as (x, y) approaches $(0, 0)$ (Figure 12.44). For example, if $m = 0$, the corresponding limit is 1 and if $m = -1$, the limit is 0. Because the function approaches different values along different paths, we conclude that the *limit does not exist*.

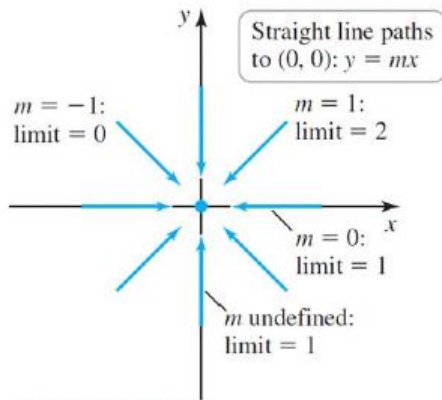


FIGURE 12.44

Question 6:

Discuss the continuity of the functions f and g . Explain any differences.

$$f(x, y) = \begin{cases} \frac{x^2 + 2xy^2 + y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$$g(x, y) = \begin{cases} \frac{x^2 + 2xy^2 + y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 1, & (x, y) = (0, 0) \end{cases}$$

1. $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} \left(\frac{x^2 + 2xy^2 + y^2}{x^2 + y^2} \right)$
2. $= \lim_{(x,y) \rightarrow (0,0)} \left(1 + \frac{2xy^2}{x^2 + y^2} \right)$
3. $= 1$
4. (same limit for g)
5. Thus, f is not continuous at $(0, 0)$, whereas g is continuous at $(0, 0)$.

Question 7:

Find both first partial derivatives.

$$z = \ln(x^2 + y^2)$$

1. $\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}$
2. $\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$

Question 8:

Evaluate f_x and f_y at the given point.

$$f(x, y) = \frac{xy}{x - y}, \quad (2, -2)$$

1. $f_x(x, y) = \frac{y(x - y) - xy}{(x - y)^2}$
2. $= \frac{-y^2}{(x - y)^2}$
3. At $(2, -2)$: $f_x(2, -2) = -\frac{1}{4}$
4. $f_y(x, y) = \frac{x(x - y) + xy}{(x - y)^2}$
5. $= \frac{x^2}{(x - y)^2}$
6. At $(2, -2)$: $f_y(2, -2) = \frac{1}{4}$

Question : 9

Find dw/dt (a) using the appropriate Chain Rule and (b) by converting w to a function of t before differentiating.

$$w = xy + xz + yz, \quad x = t - 1, \quad y = t^2 - 1, \quad z = t$$

1. (a) $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$
2. $= (y + z) + (x + z)(2t) + (x + y)$
3. $= (t^2 - 1 + t) + (t - 1 + t)(2t) + (t - 1 + t^2 - 1)$
4. $= 3(2t^2 - 1)$
5. (b) $w = (t - 1)(t^2 - 1) + (t - 1)t + (t^2 - 1)t$
6. $\frac{dw}{dt} = 2t(t - 1) + (t^2 - 1) + 2t - 1 + 3t^2 - 1$
7. $= 3(2t^2 - 1)$

Question 9

Find $\partial w/\partial s$ and $\partial w/\partial t$ when $s = 1$ and $t = 2\pi$ for the function given by

$$w = xy + yz + xz$$

where $x = s \cos t$, $y = s \sin t$, and $z = t$.

Solution

$$\begin{aligned} \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s} \\ &= (y+z)(\cos t) + (x+z)(\sin t) + (y+x)(0) \\ &= (y+z)(\cos t) + (x+z)(\sin t). \end{aligned}$$

When $s = 1$ and $t = 2\pi$, you have $x = 1$, $y = 0$, and $z = 2$.
 $\partial w/\partial s = (0 + 2\pi)(1) + (1 + 2\pi)(0) = 2\pi$. Furthermore,

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} \\ &= (y+z)(-s \sin t) + (x+z)(s \cos t) + (y+x)(1) \end{aligned}$$

and for $s = 1$ and $t = 2\pi$, it follows that

$$\begin{aligned} \frac{\partial w}{\partial t} &= (0 + 2\pi)(0) + (1 + 2\pi)(1) + (0 + 1)(1) \\ &= 2 + 2\pi. \end{aligned}$$

Question : 10

$$w = e^{xyz}, \quad x = 3u + v, \quad y = 3u - v, \quad z = u^2v$$

Use appropriate forms of the chain rule to find $\partial w/\partial u$ and $\partial w/\partial v$.

Solution

$$\frac{\partial w}{\partial u} = yze^{xyz}(3) + xze^{xyz}(3) + xye^{xyz}(2uv) = e^{xyz}(3yz + 3xz + 2xyuv)$$

$$\frac{\partial w}{\partial v} = yze^{xyz}(1) + xze^{xyz}(-1) + xye^{xyz}(u^2) = e^{xyz}(yz - xz + xyu^2)$$

Question 11

Differentiate implicitly to find the first partial derivatives of z .

$$x^2 + 2yz + z^2 = 1$$

1. $F(x, y, z) = x^2 + 2yz + z^2 - 1$

2. $= 0$

3. $\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}$

4. $= \frac{-2x}{2y + 2z}$

5. $= \frac{-x}{y + z}$

6. $\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$

7. $= \frac{-2z}{2y + 2z}$

8. $= \frac{-z}{y + z}$

Question 12

Find the four second partial derivatives. Observe that the second mixed partials are equal.

$$z = \sqrt{x^2 + y^2}$$

1. $\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$

2. $\frac{\partial^2 z}{\partial x^2} = \frac{y^2}{(x^2 + y^2)^{3/2}}$

3. $\frac{\partial^2 z}{\partial y \partial x} = \frac{-xy}{(x^2 + y^2)^{3/2}}$

4. $\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$

5. $\frac{\partial^2 z}{\partial y^2} = \frac{x^2}{(x^2 + y^2)^{3/2}}$

6. $\frac{\partial^2 z}{\partial x \partial y} = \frac{-xy}{(x^2 + y^2)^{3/2}}$

Question 13

Differentiate implicitly to find dy/dx .

$$x^2 - 3xy + y^2 - 2x + y - 5 = 0$$

- $\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$
- $= -\frac{2x - 3y - 2}{-3x + 2y + 1}$
- $= \frac{3y - 2x + 2}{2y - 3x + 1}$

Question 14

For $f(x, y)$, find all values of x and y such that $f_x(x, y) = 0$ and $f_y(x, y) = 0$ simultaneously.

$$f(x, y) = x^2 + 4xy + y^2 - 4x + 16y + 3$$

- $f_x(x, y) = 2x + 4y - 4$
- $f_y(x, y) = 4x + 2y + 16$
- $f_x = f_y = 0$:
- $2x + 4y = 4$
- $4x + 2y = -16$
- Solving for x and y , $x = -6$ and
- $y = 4$.

Question 15

Find the four second partial derivatives of

$$f(x, y) = 3x^4y - 2xy + 5xy^3.$$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(3x^4y - 2xy + 5xy^3) = 12x^3y - 2y + 5y^3$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(3x^4y - 2xy + 5xy^3) = 3x^4 - 2x + 15xy^2.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(12x^3y - 2y + 5y^3) = 36x^2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^4 - 2x + 15xy^2) = 30xy$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^4 - 2x + 15xy^2) = 12x^3 - 2 + 15y^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(12x^3y - 2y + 5y^3) = 12x^3 - 2 + 15y^2.$$

Question 16

Show that $f_{xz} = f_{zx}$ and $f_{xzz} = f_{zxx} = f_{zzx}$ for the function given by

$$f(x, y, z) = ye^x + x \ln z.$$

Solution

First partials:

$$f_x(x, y, z) = ye^x + \ln z, \quad f_z(x, y, z) = \frac{x}{z}$$

Second partials (note that the first two are equal):

$$f_{xz}(x, y, z) = \frac{1}{z}, \quad f_{zx}(x, y, z) = \frac{1}{z}, \quad f_{zz}(x, y, z) = -\frac{x}{z^2}$$

Third partials (note that all three are equal):

$$f_{xzz}(x, y, z) = -\frac{1}{z^2}, \quad f_{zxx}(x, y, z) = -\frac{1}{z^2}, \quad f_{zzx}(x, y, z) = -\frac{1}{z^2}$$

Question 17

Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Solution

$$f_y = -4xyz + x^2$$

$$f_{yx} = -4yz + 2x$$

$$f_{yxy} = -4z$$

$$f_{yxyz} = -4$$