MATH152 CALCULUS II TUTORIAL – VI

(13.11.2015)

Question 1:

Find $\|\mathbf{r}(t)\|$.

$\mathbf{r}(t) = \sin 3t\mathbf{i} + \cos 3t\mathbf{j} + t\mathbf{k}$

1. $\|\mathbf{r}(t)\| = \sqrt{(\sin 3t)^2 + (\cos 3t)^2 + t^2}$
2. $= \sqrt{1 + t^2}$

Question 2 :

Find $\mathbf{r}(t) \cdot \mathbf{u}(t)$. Is the result a vector-valued function? Explain.

$\mathbf{r}(t) = (3t - 1)\mathbf{i} + \frac{1}{4}t^3\mathbf{j} + 4\mathbf{k}$

$\mathbf{u}(t) = t^2\mathbf{i} - 8\mathbf{j} + t^3\mathbf{k}$

1. $\mathbf{r}(t) \cdot \mathbf{u}(t) = (3t - 1)(t^2) + \left(\frac{1}{4}t^3\right)(-8) + 4(t^3)$
2. $= 3t^3 - t^2 - 2t^3 + 4t^3$
3. $= 5t^3 - t^2$, a scalar.
4. The dot product is a scalar-valued function.

Question 3 :

Represent the plane curve by a vector-valued function. (There are many correct answers.)

$x^2 + y^2 = 25$

1. Let $x = 5 \cos t$,
2. then $y = 5 \sin t$.
3. $\mathbf{r}(t) = 5 \cos t\mathbf{i} + 5 \sin t\mathbf{j}$

Question 4:

Compute the derivative of the following functions.

Let $\mathbf{r}(t) = t^2\mathbf{i} + e^t\mathbf{j} - (2\cos \pi t)\mathbf{k}$. Then

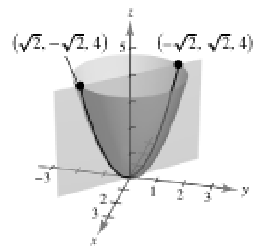
$$\mathbf{r}'(t) = \frac{d}{dt}(t^2)\mathbf{i} + \frac{d}{dt}(e^t)\mathbf{j} - \frac{d}{dt}(2\cos \pi t)\mathbf{k}$$

$$= 2t\mathbf{i} + e^t\mathbf{j} + (2\pi \sin \pi t)\mathbf{k}$$ ◄

Question 5:

Sketch the space curve represented by the intersection of the surfaces. Then represent the curve by a vector-valued function using the given parameter.

Surfaces	*Parameter*
$z = x^2 + y^2, \quad x + y = 0$	$x = t$

1. 

2. Let $x = t$,
3. then $y = -x$
4. $= -t$ and
5. $z = x^2 + y^2$
6. $= 2t^2$. Therefore,
7. $x = t$,
8. $y = -t$,
9. $z = 2t^2$.
10. $\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$

Question 6:

Evaluate the integral

$$\int_0^1 \mathbf{r}(t)\,dt = \int_0^1 \left(\sqrt[3]{t}\,\mathbf{i} + \frac{1}{t+1}\mathbf{j} + e^{-t}\mathbf{k}\right) dt.$$

Solution

$$\int_0^1 \mathbf{r}(t)\,dt = \left(\int_0^1 t^{1/3}\,dt\right)\mathbf{i} + \left(\int_0^1 \frac{1}{t+1}\,dt\right)\mathbf{j} + \left(\int_0^1 e^{-t}\,dt\right)\mathbf{k}$$

$$= \left[\left(\frac{3}{4}\right)t^{4/3}\right]_0^1 \mathbf{i} + \left[\ln|t+1|\right]_0^1 \mathbf{j} + \left[-e^{-t}\right]_0^1 \mathbf{k}$$

$$= \frac{3}{4}\mathbf{i} + (\ln 2)\mathbf{j} + \left(1 - \frac{1}{e}\right)\mathbf{k}$$

Question 7:

Evaluate the limit.

$$\lim_{t\to 2}\left(t\mathbf{i} + \frac{t^2-4}{t^2-2t}\mathbf{j} + \frac{1}{t}\mathbf{k}\right)$$

1. $\lim_{t\to 2}\left[t\mathbf{i} + \frac{t^2-4}{t^2-2t}\mathbf{j} + \frac{1}{t}\mathbf{k}\right] = 2\mathbf{i} + 2\mathbf{j} + \frac{1}{2}\mathbf{k}$
2. since

$$\lim_{t\to 2}\frac{t^2-4}{t^2-2t} = \lim_{t\to 2}\frac{2t}{2t-2}$$

3. $= 2.$ (L'Hôpital's Rule)

Question : 8

Find the unit tangent vector to the curve given by

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$$

when $t = 1$.

Solution The derivative of $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}. \qquad \text{Derivative of } \mathbf{r}(t)$$

So, the unit tangent vector is

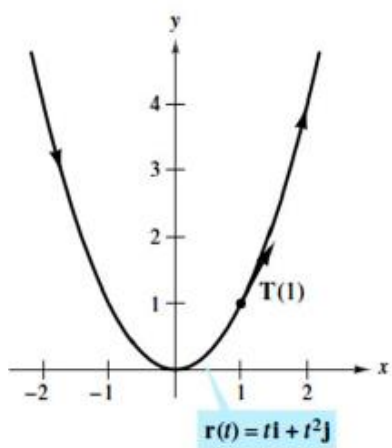
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \qquad \text{Definition of } \mathbf{T}(t)$$

$$= \frac{1}{\sqrt{1+4t^2}}(\mathbf{i} + 2t\mathbf{j}). \qquad \text{Substitute for } \mathbf{r}'(t).$$

When $t = 1$, the unit tangent vector is

$$\mathbf{T}(1) = \frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$$

as shown in Figure 12.20.



The direction of the unit tangent vector depends on the orientation of the curve.
Figure 12.20

Question : 9

Find the arc length of that portion of the circular helix

$$x = \cos t, \quad y = \sin t, \quad z = t$$

from $t = 0$ to $t = \pi$.

Solution

Set $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k} = \langle \cos t, \sin t, t \rangle$. Then

$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$ and $\|\mathbf{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} =$

the arc length of the helix is

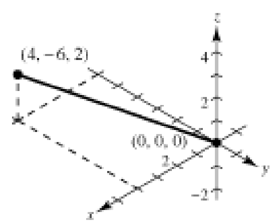
$$L = \int_0^{\pi} \left\| \frac{d\mathbf{r}}{dt} \right\| dt = \int_0^{\pi} \sqrt{2}\, dt = \sqrt{2}\pi$$

Question 10

Sketch the space curve and find its length over the given interval.

Function	*Interval*
$\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{j} + t\mathbf{k}$	$[0, 2]$

1.



2. $\dfrac{dx}{dt} = 2$

3. $\dfrac{dy}{dt} = -3$

4. $\dfrac{dz}{dt} = 1$

5. $s = \displaystyle\int_0^2 \sqrt{2^2 + (-3)^2 + 1^2}\, dt$

6. $= \displaystyle\int_0^2 \sqrt{14}\, dt$

7. $= \left[\sqrt{14}\, t \right]_0^2$

8. $= 2\sqrt{14}$

Question 11

Find and simplify the function values.

$f(x, y) = xe^y$

(a) $(5, 0)$ (b) $(3, 2)$ (c) $(2, -1)$

(d) $(5, y)$ (e) $(x, 2)$ (f) (t, t)

1. (a) $f(5, 0) = 5e^0$
2. $= 5$
3. (b) $f(3, 2) = 3e^2$
4. (c) $f(2, -1) = 2e^{-1}$
5. $= \dfrac{2}{e}$
6. (d) $f(5, y) = 5e^y$
7. (e) $f(x, 2) = xe^2$
8. (f) $f(t, t) = te^t$

Question 12

Describe the domain and range of the function.

$f(x, y) = \ln(4 - x - y)$

1. Domain: $4 - x - y > 0$
2. $x + y < 4$
3. $\{(x, y): y < -x + 4\}$
4. Range: all real numbers

Question 13

Describe the domain and range of the function.

$f(x, y) = \sqrt{4 - x^2 - y^2}$

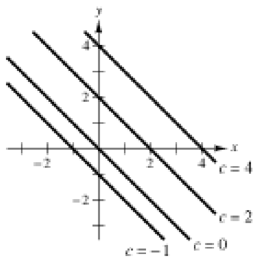
1. Domain: $4 - x^2 - y^2 \geq 0$
2. $x^2 + y^2 \leq 4$
3. $\{(x, y): x^2 + y^2 \leq 4\}$
4. Range: $0 \leq z \leq 2$

Question 14

Describe the level curves of the function. Sketch the level curves for the given c-values.

$z = x + y, \quad c = -1, 0, 2, 4$

1. Level curves are parallel lines of the form $x + y = c$.

2. 

Question 15

Describe the level curves of the function. Sketch the level curves for the given c-values.

$z = \sqrt{25 - x^2 - y^2}, \quad c = 0, 1, 2, 3, 4, 5$

1. The level curves are of the form

$$c = \sqrt{25 - x^2 - y^2},$$

2. $x^2 + y^2 = 25 - c^2$.

3. Thus, the level curves are circles of radius 5 or less, centered at the origin.

4.