MATH152 CALCULUS II TUTORIAL – V

(06.11.2015)

Question 1:

Find an equation of the plane passing through the point perpendicular to the given vector or line.

Point	*Perpendicular to*
$(3, 2, 2)$	$\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

1. Normal vector: $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
2. $2(x - 3) + 3(y - 2) - 1(z - 2) = 0$
3. $2x + 3y - z = 10$

Question 2 :

Find an equation of the plane.

The plane contains the lines given by

$$\frac{x-1}{-2} = y - 4 = z \text{ and } \frac{x-2}{-3} = \frac{y-1}{4} = \frac{z-2}{-1}.$$

1. The direction vectors for the lines are $\mathbf{u} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$,
2. $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.
3. Normal vector: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ -3 & 4 & -1 \end{vmatrix}$
4. $= -5(\mathbf{i} + \mathbf{j} + \mathbf{k})$
5. Point of intersection of the lines: $(-1, 5, 1)$
6. $(x + 1) + (y - 5) + (z - 1) = 0$
7. $x + y + z = 5$

Question 3 :

Find the distance between the point and the line given by the set of parametric equations.

$(1, 5, -2)$: $x = 4t - 2$, $y = 3$, $z = -t + 1$

1. $\mathbf{u} = \langle 4, 0, -1 \rangle$ is the direction vector for the line.
2. $Q(1, 5, -2)$ is the given point,
3. and $P(-2, 3, 1)$ is on the line.
4. Hence, $\overrightarrow{PQ} = \langle 3, 2, -3 \rangle$ and
5. $\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -3 \\ 4 & 0 & -1 \end{vmatrix}$
6. $= \langle -2, -9, -8 \rangle$.
7. $D = \dfrac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|}$
8. $= \dfrac{\sqrt{149}}{\sqrt{17}}$
9. $= \dfrac{\sqrt{2533}}{17}$

Question 4:

Find an equation of the plane.

The plane passes through the points $(2, 2, 1)$ and $(-1, 1, -1)$ and is perpendicular to the plane $2x - 3y + z = 3$.

1. Let $\mathbf{v}$ be the vector from $(-1, 1, -1)$ to $(2, 2, 1)$:

   $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
2. Let $\mathbf{n}$ be a vector normal to the plane $2x - 3y + z = 3$:

   $\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
3. Since $\mathbf{v}$ and $\mathbf{n}$ both lie in the plane P, the normal vector to P is

   $\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 2 & -3 & 1 \end{vmatrix}$
4. $= 7\mathbf{i} + \mathbf{j} - 11\mathbf{k}$

Question 5:

Find the distance between the point and the plane.

$(2, 8, 4)$, $2x + y + z = 5$

1. Point: $Q(2, 8, 4)$
2. Plane: $2x + y + z = 5$
3. Normal to plane: $\mathbf{n} = \langle 2, 1, 1\rangle$
4. Point in plane: $P\langle 0, 0, 5\rangle$
5. Vector: $\overrightarrow{PQ} = \langle 2, 8, -1\rangle$
6. $D = \dfrac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$
7. $= \dfrac{11}{\sqrt{6}}$
8. $= \dfrac{11\sqrt{6}}{6}$

Question 6:

Determine whether the planes are parallel, orthogonal, or neither. If they are neither parallel nor orthogonal, find the angle of intersection.

$5x - 3y + z = 4$

$x + 4y + 7z = 1$

1. The normal vectors to the planes are

   $\mathbf{n}_1 = \langle 5, -3, 1\rangle$,
2. $\mathbf{n}_2 = \langle 1, 4, 7\rangle$,
3. $\cos\theta = \dfrac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}$
4. $= 0.$
5. Thus, $\theta = \dfrac{\pi}{2}$ and the planes are orthogonal.

Question 7:

A plane through three points Find an equation of the plane that passes through the (noncollinear) points $P(2, -1, 3)$, $Q(1, 4, 0)$, and $R(0, -1, 5)$.

SOLUTION To write an equation for the plane, we must find a normal vector. Because P, Q, and R lie in the plane, the vectors $\overrightarrow{PQ} = \langle -1, 5, -3\rangle$ and $\overrightarrow{PR} = \langle -2, 0, 2\rangle$ also lie in the plane. The cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$; therefore a vector normal to the plane is

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 5 & -3 \\ -2 & 0 & 2 \end{vmatrix} = 10\mathbf{i} + 8\mathbf{j} + 10\mathbf{k}.$$

Any scalar multiple of $\mathbf{n}$ may be used as the normal vector. Choosing $\mathbf{n} = \langle 10, 8, 10\rangle$ and $P_0(2, -1, 3)$ as the fixed point in the plane an equation of the plane is

$$10(x - 2) + 8(y - (-1)) + 10(z - 3) = 0 \quad \text{or} \quad 5x + 4y + 5z = 21.$$

Using either Q or R as the fixed point in the plane leads to an equivalent equation of the plane.

Question : 8

Parallel planes Find an equation of the plane Q that passes through the point $(-2, 4, 1)$ and is parallel to the plane R: $3x - 2y + z = 4$.

SOLUTION The vector $\mathbf{n} = \langle 3, -2, 1\rangle$ is normal to R. Because Q and R are parallel, $\mathbf{n}$ is also normal to Q. Therefore, an equation of Q passing through $(-2, 4, 1)$ with normal vector $\langle 3, -2, 1\rangle$ is

$$3(x + 2) - 2(y - 4) + (z - 1) = 0 \quad \text{or} \quad 3x - 2y + z = -13.$$

Question : 9

Intersecting planes Find an equation of the line of intersection of the planes Q: $x + 2y + z = 5$ and R: $2x + y - z = 7$.

SOLUTION First note that the vectors normal to the planes, $\mathbf{n}_Q = \langle 1, 2, 1\rangle$ and $\mathbf{n}_R = \langle 2, 1, -1\rangle$, are *not* multiples of each other. Therefore, the planes are not parallel and they must intersect in a line; call it ℓ. To find an equation of ℓ, we need two pieces of information: a point on ℓ and a vector pointing in the direction of ℓ. Here is one of several ways to find a point on ℓ. Setting $z = 0$ in the equations of the planes gives equations of the lines in which the planes intersect the xy-plane:

$$x + 2y = 5$$
$$2x + y = 7$$

Solving these equations simultaneously, we find that $x = 3$ and $y = 1$. Combinin
result with $z = 0$, we see that $(3, 1, 0)$ is a point on ℓ

We next find a vector parallel to ℓ. Because ℓ lies in Q and R, it is orthogona normal vectors $\mathbf{n}_Q$ and $\mathbf{n}_R$. Therefore, the cross product of $\mathbf{n}_Q$ and $\mathbf{n}_R$ is a vector p to ℓ (Figure 12.7). In this case, the cross product is

$$\mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} = \langle -3, 3, -3 \rangle.$$

An equation of the line ℓ in the direction of the vector $\langle -3, 3, -3\rangle$ passing throug point $(3, 1, 0)$ is

$$\begin{aligned} \mathbf{r}(t) &= \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle \\ &= \langle 3, 1, 0 \rangle + t\langle -3, 3, -3 \rangle \\ &= \langle 3 - 3t, 1 + 3t, -3t \rangle, \end{aligned}$$

where $-\infty < t < \infty$. You can check that any point (x, y, z) with $x = 3 - 3t$, $y = 1 + 3t$, and $z = -3t$ satisfies the equations of both planes.

Question : 10

Find the intersection of the line and plane

$x = 3 + 8t, \quad y = 4 + 5t, \quad z = -3 - t$

$x - 3y + 5z = 12.$

Solution. If we let (x_0, y_0, z_0) be the point of intersection, then the coordinates point satisfy both the equation of the plane and the parametric equations of the line.

$$x_0 - 3y_0 + 5z_0 = 12$$

and for some value of t, say $t = t_0$,

$$x_0 = 3 + 8t_0, \quad y_0 = 4 + 5t_0, \quad z_0 = -3 - t_0$$

Substituting (8) in (7) yields

$$(3 + 8t_0) - 3(4 + 5t_0) + 5(-3 - t_0) = 12$$

Solving for t_0 yields $t_0 = -3$ and on substituting this value in (8), we obtain

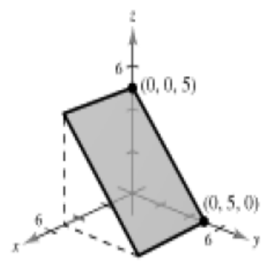
$$(x_0, y_0, z_0) = (-21, -11, 0)$$ ◀

Question 11

Label any intercepts and sketch a graph of the plane.

$y + z = 5$

1.



Question 12

Find the domain of the vector-valued function.

$\mathbf{r}(t) = \ln t\mathbf{i} - e^t\mathbf{j} - t\mathbf{k}$

1. Component functions: $f(t) = \ln t$
2. $g(t) = -e^t$
3. $h(t) = -t$
4. Domain: $(0, \infty)$

Question 13

Evaluate (if possible) the vector-valued function at each given value of t.

$\mathbf{r}(t) = \ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + 3t\mathbf{k}$

(a) $\mathbf{r}(2)$ (b) $\mathbf{r}(-3)$ (c) $\mathbf{r}(t - 4)$

Solution.

1. (a) $\mathbf{r}(2) = \ln 2\mathbf{i} + \frac{1}{2}\mathbf{j} + 6\mathbf{k}$

2. (b) $\mathbf{r}(-3)$ is not defined. ($\ln(-3)$ does not exist.)

3. (c) $\mathbf{r}(t-4) = \ln(t-4)\mathbf{i} + \frac{1}{t-4}\mathbf{j} + 3(t-4)\mathbf{k}$

Question 14

Sketch the curve represented by the vector-valued function and give the orientation of the curve.

$\mathbf{r}(t) = (-t+1)\mathbf{i} + (4t+2)\mathbf{j} + (2t+3)\mathbf{k}$

1. $x = -t + 1$
2. $y = 4t + 2$
3. $z = 2t + 3$
4. Line passing through the points:

   $(0, 6, 5)$, $(1, 2, 3)$

5.