MATH152 CALCULUS II TUTORIAL – V

(27.10.2017)

Question 1:

Find a set of parametric equations of the line.

The line passes through the point $(2, 3, 4)$ and is perpendicular to the plane given by $3x + 2y - z = 6$.

1. Point: $(2, 3, 4)$
2. Direction vector: $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
3. Direction numbers: $3, 2, -1$
4. Parametric: $x = 2 + 3t, y = 3 + 2t, z = 4 - t$

Question 2:

Determine whether the lines intersect, and if so, find the point of intersection and the cosine of the angle of intersection.

$x = 4t + 2,\ y = 3,\ z = -t + 1$

$x = 2s + 2,\ y = 2s + 3,\ z = s + 1$

1. At the point of intersection, the coordinates for one line equal the corresponding coordinates for the other line. Thus,
2. (i) $4t + 2 = 2s + 2$,
3. (ii) $3 = 2s + 3$,
4. and (iii) $-t + 1 = s + 1$.
5. From (ii), we find that $s = 0$ and consequently,
6. from (iii), $t = 0$.
7. Letting $s = t = 0$, we see that equation (i) is satisfied and therefore the two lines intersect.
8. Substituting zero for s or for t, we obtain the point $(2, 3, 1)$.
9. $\mathbf{u} = 4\mathbf{i} - \mathbf{k}$ (First line)
10. $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ (Second line)
11. $\cos\theta = \dfrac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$
12. $= \dfrac{8 - 1}{\sqrt{17}\sqrt{9}}$
13. $= \dfrac{7}{3\sqrt{17}}$
14. $= \dfrac{7\sqrt{17}}{51}$

Question 3 :

Find an equation of the plane.

The plane passes through $(-3, -4, 2)$, $(-3, 4, 1)$, and $(1, 1, -2)$.

1. $P = (-3, -4, 2)$
2. $Q = (-3, 4, 1)$
3. $R = (1, 1, -2)$
4. $\overrightarrow{PQ} = \langle 0, 8, -1\rangle$
5. $\overrightarrow{PR} = \langle 4, 5, -4\rangle$
6. $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$
7. $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 8 & -1 \\ 4 & 5 & -4 \end{vmatrix}$
8. $= -27\mathbf{i} - 4\mathbf{j} - 32\mathbf{k}$
9. $-27(x + 3) - 4(y + 4) - 32(z - 2) = 0$
10. $27x + 4y + 32z = -33$

Question 4 :

Find an equation of the plane.

The plane contains the lines given by

$\dfrac{x - 1}{-2} = y - 4 = z$ and $\dfrac{x - 2}{-3} = \dfrac{y - 1}{4} = \dfrac{z - 2}{-1}$.

1. The direction vectors for the lines are $\mathbf{u} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$,
2. $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.
3. Normal vector: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ -3 & 4 & -1 \end{vmatrix}$
4. $= -5(\mathbf{i} + \mathbf{j} + \mathbf{k})$
5. Point of intersection of the lines: $(-1, 5, 1)$
6. $(x + 1) + (y - 5) + (z - 1) = 0$
7. $x + y + z = 5$

Question 5 :

Find an equation of the plane.

The plane passes through the points $(1, -2, -1)$ and $(2, 5, 6)$ and is parallel to the x-axis.

1. Let $\mathbf{u} = \mathbf{i}$ and
2. let $\mathbf{v}$ be the vector from $(1, -2, -1)$ to $(2, 5, 6)$:

   $\mathbf{v} = \mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$
3. Since $\mathbf{u}$ and $\mathbf{v}$ both lie in the plane P, the normal vector to P is:

   $$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 7 & 7 \end{vmatrix}$$
4. $= -7\mathbf{j} + 7\mathbf{k}$
5. $= -7(\mathbf{j} - \mathbf{k})$
6. $[y - (-2)] - [z - (-1)] = 0$
7. $y - z = -1$

Question 6:

Find an equation of the plane.

The plane passes through the points $(2, 2, 1)$ and $(-1, 1, -1)$ and is perpendicular to the plane $2x - 3y + z = 3$.

1. Let $\mathbf{v}$ be the vector from $(-1, 1, -1)$ to $(2, 2, 1)$:

   $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
2. Let $\mathbf{n}$ be a vector normal to the plane $2x - 3y + z = 3$:

   $\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
3. Since $\mathbf{v}$ and $\mathbf{n}$ both lie in the plane P, the normal vector to P is

   $$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 2 & -3 & 1 \end{vmatrix}$$
4. $= 7\mathbf{i} + \mathbf{j} - 11\mathbf{k}$

Question 7:

Find the distance between the point and the plane.

$(0, 0, 0)$, $2x + 3y + z = 12$

1. Point: $Q(0, 0, 0)$
2. Plane: $2x + 3y + z - 12 = 0$
3. Normal to plane: $\mathbf{n} = \langle 2, 3, 1 \rangle$
4. Point in plane: $P(6, 0, 0)$
5. Vector: $\overrightarrow{PQ} = \langle -6, 0\ 0 \rangle$
6. $D = \dfrac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$
7. $= \dfrac{|-12|}{\sqrt{14}}$
8. $= \dfrac{6\sqrt{14}}{7}$

Question 8:

Find the point(s) of intersection (if any) of the plane and the line. Also determine whether the line lies in the plane.

$2x + 3y = 10$, $\dfrac{x - 1}{3} = \dfrac{y + 1}{-2} = z - 3$

1. Writing the equation of the line in parametric form and substituting into the equation of the plane we have:
2. $x = 1 + 3t$
3. $y = -1 - 2t$
4. $z = 3 + t$
5. $2(1 + 3t) + 3(-1 - 2t) = 10$
6. $-1 = 10$, contradiction
7. Therefore, the line does not intersect the plane.

Question 9:

Find a set of parametric equations for the line of intersection of the planes.

$3x + 2y - z = 7$

$x - 4y + 2z = 0$

1. The normals to the planes are $\mathbf{n}_1 = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and
2. $\mathbf{n}_2 = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.
3. The direction vector for the line is

$$\mathbf{n}_2 \times \mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 2 \\ 3 & 2 & -1 \end{vmatrix}$$

4. $= 7(\mathbf{j} + 2\mathbf{k})$.
5. Now find a point of intersection of the planes.

$6x + 4y - 2z = 14$

6. $x - 4y + 2z = 0$
7. $7x = 14$
8. $x = 2$
9. Substituting 2 for x in the second equation, we have $-4y + 2z = -2$ or $z = 2y - 1$.
10. Letting $y = 1$, a point of intersection is $(2, 1, 1)$.
11. $x = 2$
12. $y = 1 + t$
13. $z = 1 + 2t$

Question 10

Find the domain of the vector-valued function.

$\mathbf{r}(t) = \ln t\mathbf{i} - e^t\mathbf{j} - t\mathbf{k}$

1. Component functions: $f(t) = \ln t$
2. $g(t) = -e^t$
3. $h(t) = -t$
4. Domain: $(0, \infty)$

Question 11

Evaluate (if possible) the vector-valued function at each given value of t.

$\mathbf{r}(t) = \ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + 3t\mathbf{k}$

(a) $\mathbf{r}(2)$ (b) $\mathbf{r}(-3)$ (c) $\mathbf{r}(t-4)$

Solution.

1. (a) $\mathbf{r}(2) = \ln 2\mathbf{i} + \frac{1}{2}\mathbf{j} + 6\mathbf{k}$
2. (b) $\mathbf{r}(-3)$ is not defined. ($\ln(-3)$ does not exist.)
3. (c) $\mathbf{r}(t-4) = \ln(t-4)\mathbf{i} + \frac{1}{t-4}\mathbf{j} + 3(t-4)\mathbf{k}$