MATH152 CALCULUS II TUTORIAL – V

(28.10.2016)

Question 1:

Find an equation of the plane.

The plane passes through $(0, 0, 0)$, $(1, 2, 3)$, and $(-2, 3, 3)$.

1. Let $\mathbf{u}$ be the vector from $(0, 0, 0)$ to $(1, 2, 3)$:

   $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
2. Let $\mathbf{v}$ be the vector from $(0, 0, 0)$ to $(-2, 3, 3)$:

   $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$
3. Normal vector: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & 3 & 3 \end{vmatrix}$
4. $= -3\mathbf{i} + (-9)\mathbf{j} + 7\mathbf{k}$
5. $-3(x - 0) - 9(y - 0) + 7(z - 0) = 0$
6. $3x + 9y - 7z = 0$

Question 2 :

Find an equation of the plane.

The plane contains the lines given by

$$\frac{x-1}{-2} = y - 4 = z \text{ and } \frac{x-2}{-3} = \frac{y-1}{4} = \frac{z-2}{-1}.$$

1. The direction vectors for the lines are $\mathbf{u} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$,
2. $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.
3. Normal vector: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ -3 & 4 & -1 \end{vmatrix}$
4. $= -5(\mathbf{i} + \mathbf{j} + \mathbf{k})$
5. Point of intersection of the lines: $(-1, 5, 1)$
6. $(x + 1) + (y - 5) + (z - 1) = 0$
7. $x + y + z = 5$

Question 3 :

Determine if any of the planes are parallel or identical.

P_1: $3x - 2y + 5z = 10$

P_2: $-6x + 4y - 10z = 5$

P_3: $-3x + 2y + 5z = 8$

P_4: $75x - 50y + 125z = 250$

1. P_1: $\mathbf{n} = \langle 3, -2, 5\rangle$ $(1, -1, 1)$ on plane
2. P_2: $\mathbf{n} = \langle -6, 4, -10\rangle$ $(1, -1, 1)$ not on plane
3. P_3: $\mathbf{n} = \langle -3, 2, 5\rangle$
4. P_4: $\mathbf{n} = \langle 75, -50, 125\rangle$ $(1, -1, 1)$ on plane
5. P_1 and P_4 are identical.
6. $P_1 = P_4$ is parallel to P_2.

Question 4:

Find an equation of the plane.

The plane passes through the points $(1, -2, -1)$ and $(2, 5, 6)$ and is parallel to the x-axis.

1. Let $\mathbf{u} = \mathbf{i}$ and
2. let $\mathbf{v}$ be the vector from $(1, -2, -1)$ to $(2, 5, 6)$:

   $\mathbf{v} = \mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$
3. Since $\mathbf{u}$ and $\mathbf{v}$ both lie in the plane P, the normal vector to P is:

   $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 7 & 7 \end{vmatrix}$
4. $= -7\mathbf{j} + 7\mathbf{k}$
5. $= -7(\mathbf{j} - \mathbf{k})$
6. $[y - (-2)] - [z - (-1)] = 0$
7. $y - z = -1$

Question 5:

Find an equation of the plane.

The plane passes through the points $(2, 2, 1)$ and $(-1, 1, -1)$ and is perpendicular to the plane $2x - 3y + z = 3$.

1. Let $\mathbf{v}$ be the vector from $(-1, 1, -1)$ to $(2, 2, 1)$:

$\mathbf{v} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

2. Let $\mathbf{n}$ be a vector normal to the plane $2x - 3y + z = 3$:

$\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

3. Since $\mathbf{v}$ and $\mathbf{n}$ both lie in the plane P, the normal vector to P is

$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 2 & -3 & 1 \end{vmatrix}$$

4. $= 7\mathbf{i} + \mathbf{j} - 11\mathbf{k}$

Question 6:

Find the distance between the point and the plane.

$(0, 0, 0),\ 2x + 3y + z = 12$

1. Point: $Q(0, 0, 0)$
2. Plane: $2x + 3y + z - 12 = 0$
3. Normal to plane: $\mathbf{n} = \langle 2, 3, 1 \rangle$
4. Point in plane: $P(6, 0, 0)$
5. Vector: $\overrightarrow{PQ} = \langle -6, 0\ 0 \rangle$
6. $D = \dfrac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$
7. $= \dfrac{|-12|}{\sqrt{14}}$
8. $= \dfrac{6\sqrt{14}}{7}$

Question 7:

Find the point(s) of intersection (if any) of the plane and the line. Also determine whether the line lies in the plane.

$2x - 2y + z = 12,\ x - \dfrac{1}{2} = \dfrac{y + (3/2)}{-1} = \dfrac{z + 1}{2}$

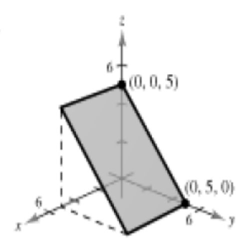
1. Writing the equation of the line in parametric form and substituting into the equation of the plane we have:
2. $x = \dfrac{1}{2} + t$
3. $y = \dfrac{-3}{2} - t$
4. $z = -1 + 2t$
5. $2\left(\dfrac{1}{2} + t\right) - 2\left(\dfrac{-3}{2} - t\right) + (-1 + 2t) = 12$
6. $t = \dfrac{3}{2}$
7. Substituting $t = 3/2$ into the parametric equations for the line we have the point of intersection $(2, -3, 2)$.
8. The line does not lie in the plane.

Question 8:

Label any intercepts and sketch a graph of the plane.

$y + z = 5$

1.



Question 9:

Find a set of parametric equations for the line of intersection of the planes.

$3x + 2y - z = 7$

$x - 4y + 2z = 0$

1. The normals to the planes are $\mathbf{n}_1 = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and
2. $\mathbf{n}_2 = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.
3. The direction vector for the line is

$$\mathbf{n}_2 \times \mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 2 \\ 3 & 2 & -1 \end{vmatrix}$$

4. $= 7(\mathbf{j} + 2\mathbf{k})$.
5. Now find a point of intersection of the planes.

$6x + 4y - 2z = 14$

6. $x - 4y + 2z = 0$
7. $7x = 14$
8. $x = 2$
9. Substituting 2 for x in the second equation, we have $-4y + 2z = -2$ or $z = 2y - 1$.
10. Letting $y = 1$, a point of intersection is $(2, 1, 1)$.
11. $x = 2$
12. $y = 1 + t$
13. $z = 1 + 2t$

Question 10

Find the domain of the vector-valued function.

$\mathbf{r}(t) = \ln t\mathbf{i} - e^t\mathbf{j} - t\mathbf{k}$

1. Component functions: $f(t) = \ln t$
2. $g(t) = -e^t$
3. $h(t) = -t$
4. Domain: $(0, \infty)$

Question 11

Evaluate (if possible) the vector-valued function at each given value of t.

$\mathbf{r}(t) = \ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + 3t\mathbf{k}$

(a) $\mathbf{r}(2)$ (b) $\mathbf{r}(-3)$ (c) $\mathbf{r}(t-4)$

Solution.

1. (a) $\mathbf{r}(2) = \ln 2\mathbf{i} + \frac{1}{2}\mathbf{j} + 6\mathbf{k}$
2. (b) $\mathbf{r}(-3)$ is not defined. ($\ln(-3)$ does not exist.)
3. (c) $\mathbf{r}(t-4) = \ln(t-4)\mathbf{i} + \frac{1}{t-4}\mathbf{j} + 3(t-4)\mathbf{k}$