

MATH152 CALCULUS II TUTORIAL – IV

(19.10.2018)

Question 1:

Complete the square to write the equation of the sphere in standard form. Find the center and radius.

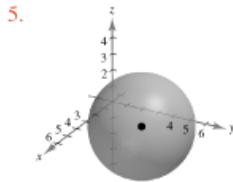
$$x^2 + y^2 + z^2 - 4x - 6y + 4 = 0$$

1. $(x^2 - 4x + 4) + (y^2 - 6y + 9) + z^2 = -4 + 4 + 9$

2. $(x - 2)^2 + (y - 3)^2 + z^2 = 9$

3. Center: $(2, 3, 0)$

4. Radius: 3



Question 3:

Find the angle θ between the vectors.

$$\mathbf{u} = \langle 1, 1, 1 \rangle, \mathbf{v} = \langle 2, 1, -1 \rangle$$

1. $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$

2. $= \frac{2}{\sqrt{3}\sqrt{6}}$

3. $= \frac{\sqrt{2}}{3}$

4. $\theta = \arccos \frac{\sqrt{2}}{3}$

5. $\approx 61.9^\circ$

Question 4:

Find the angle θ between the vectors.

$$\mathbf{u} = \langle 10, -5, 15 \rangle$$

$$\mathbf{v} = \langle -2, 1, -3 \rangle$$

1. $\mathbf{u} = -5\mathbf{v} \Rightarrow \mathbf{u}$ is parallel to \mathbf{v} and in the opposite direction.

2. $\theta = \pi$

Question 2:

Find (a) $\mathbf{u} \cdot \mathbf{v}$, (b) $\mathbf{u} \cdot \mathbf{u}$, (c) $\|\mathbf{u}\|^2$, (d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v}$, and (e) $\mathbf{u} \cdot (2\mathbf{v})$.

$$\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \quad \mathbf{v} = \mathbf{i} - \mathbf{k}$$

1. (a) $\mathbf{u} \cdot \mathbf{v} = 2(1) + (-1)(0) + 1(-1)$

2. $= 1$

3. (b) $\mathbf{u} \cdot \mathbf{u} = 2(2) + (-1)(-1) + (1)(1)$

4. $= 6$

5. (c) $\|\mathbf{u}\|^2 = 6$

6. (d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = \mathbf{v}$

7. $= \mathbf{i} - \mathbf{k}$

8. (e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) =$

9. $= 2$

Question 5:

Determine whether \mathbf{u} and \mathbf{v} are orthogonal, parallel, or neither.

$$\mathbf{u} = \langle 4, 3 \rangle, \quad \mathbf{v} = \left\langle \frac{1}{2}, -\frac{2}{3} \right\rangle$$

1. $\mathbf{u} \neq c\mathbf{v} \Rightarrow$ not parallel

2. $\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow$ orthogonal

Question 6 :

(a) Find the projection of \mathbf{u} onto \mathbf{v} , and (b) find the vector component of \mathbf{u} orthogonal to \mathbf{v} .

$$\mathbf{u} = \langle 2, 1, 2 \rangle, \quad \mathbf{v} = \langle 0, 3, 4 \rangle$$

- (a) $\mathbf{w}_1 = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$
- $= \frac{11}{25} \langle 0, 3, 4 \rangle$
- $= \left\langle 0, \frac{33}{25}, \frac{44}{25} \right\rangle$
- (b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$
- $= \left\langle 2, -\frac{8}{25}, \frac{6}{25} \right\rangle$

Question 7 :

Let $\mathbf{u} = \langle 3, -2, 1 \rangle$, $\mathbf{v} = \langle 2, -4, -3 \rangle$, and $\mathbf{w} = \langle -1, 2, 2 \rangle$.

Determine the projection of \mathbf{w} onto \mathbf{u} .

- $\text{proj}_{\mathbf{u}} \mathbf{w} = \left(\frac{\mathbf{u} \cdot \mathbf{w}}{\|\mathbf{u}\|^2} \right) \mathbf{u}$
- $= -\frac{5}{14} \langle 3, -2, 1 \rangle$
- $= \left\langle -\frac{15}{14}, \frac{10}{14}, -\frac{5}{14} \right\rangle$
- $= \left\langle -\frac{15}{14}, \frac{5}{7}, -\frac{5}{14} \right\rangle$

Question 8:

Find (a) $\mathbf{u} \times \mathbf{v}$, (b) $\mathbf{v} \times \mathbf{u}$, and (c) $\mathbf{v} \times \mathbf{v}$.

$$\mathbf{u} = \langle 7, 3, 2 \rangle$$

$$\mathbf{v} = \langle 1, -1, 5 \rangle$$

- (a) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & 2 \\ 1 & -1 & 5 \end{vmatrix}$
- $= \langle 17, -33, -10 \rangle$
- (b) $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
- $= \langle -17, 33, 10 \rangle$
- (c) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

Question 9:

Find $\mathbf{u} \times \mathbf{v}$ and show that it is orthogonal to both \mathbf{u} and \mathbf{v} .

$$\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

- $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$
- $= -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
- $= \langle -2, 3, -1 \rangle$
- $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 1(-2) + 1(3) + 1(-1)$
- $= 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$
- $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 2(-2) + 1(3) + (-1)(-1)$
- $= 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$
- $(-\mathbf{v}) \times \mathbf{u} = -(\mathbf{v} \times \mathbf{u})$
- $= \mathbf{u} \times \mathbf{v}$

Question 10:

Find the area of the triangle with the given vertices. (Hint: $\frac{1}{2}\|\mathbf{u} \times \mathbf{v}\|$ is the area of the triangle having \mathbf{u} and \mathbf{v} as adjacent sides.)

$$(2, -7, 3), (-1, 5, 8), (4, 6, -1)$$

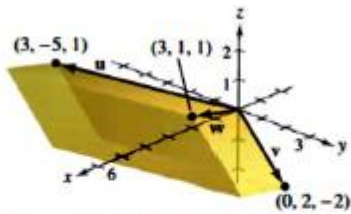
- $A(2, -7, 3), B(-1, 5, 8), C(4, 6, -1)$
- $\overrightarrow{AB} = \langle -3, 12, 5 \rangle$
- $\overrightarrow{AC} = \langle 2, 13, -4 \rangle$
- $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 12 & 5 \\ 2 & 13 & -4 \end{vmatrix}$
- $= \langle -113, -2, -63 \rangle$
- Area $= \frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\|$
- $= \frac{1}{2} \sqrt{16,742}$

Question : 11

Find the volume of the parallelepiped shown in Figure 11.42 having $\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ as adjacent edges.

Solution By Theorem 11.10, you have

$$\begin{aligned} V &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| && \text{Triple scalar product} \\ &= \begin{vmatrix} 3 & -5 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix} \\ &= 3 \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} - (-5) \begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} + (1) \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \\ &= 3(4) + 5(6) + 1(-6) \\ &= 36. \end{aligned}$$



The parallelepiped has a volume of 36.
Figure 11.42

Question : 12

Find a set of parametric equations of the line.

The line passes through the point $(2, 1, 2)$ and is parallel to the line $x = -t$, $y = 1 + t$, $z = -2 + t$.

1. Point: $(2, 1, 2)$
2. Direction vector: $\langle -1, 1, 1 \rangle$
3. Direction numbers: $-1, 1, 1$
4. Parametric: $x = 2 - t$, $y = 1 + t$, $z = 2 + t$

Question : 13

Determine if any of the lines are parallel or identical.

$$L_1: x = 6 - 3t, y = -2 + 2t, z = 5 + 4t$$

$$L_2: x = 6t, y = 2 - 4t, z = 13 - 8t$$

$$L_3: x = 10 - 6t, y = 3 + 4t, z = 7 + 8t$$

$$L_4: x = -4 + 6t, y = 3 + 4t, z = 5 - 6t$$

1. $L_1: \mathbf{v} = \langle -3, 2, 4 \rangle$ $(6, -2, 5)$ on line
2. $L_2: \mathbf{v} = \langle 6, -4, -8 \rangle$ $(6, -2, 5)$ on line
3. $L_3: \mathbf{v} = \langle -6, 4, 8 \rangle$ $(6, -2, 5)$ not on line
4. $L_4: \mathbf{v} = \langle 6, 4, -6 \rangle$ not parallel to L_1, L_2 , nor L_3
5. Hence, L_1 and L_2 are identical.
6. $L_1 = L_2$ and L_3 are parallel.

Question 14

Determine whether the lines intersect, and if so, find the point of intersection and the cosine of the angle of intersection.

$$x = 4t + 2, y = 3, z = -t + 1$$

$$x = 2s + 2, y = 2s + 3, z = s + 1$$

1. At the point of intersection, the coordinates for one line equal the corresponding coordinates for the other line. Thus,
2. (i) $4t + 2 = 2s + 2$,
3. (ii) $3 = 2s + 3$,
4. and (iii) $-t + 1 = s + 1$.
5. From (ii), we find that $s = 0$ and consequently,
6. from (iii), $t = 0$.
7. Letting $s = t = 0$, we see that equation (i) is satisfied and therefore the two lines intersect.
8. Substituting zero for s or for t , we obtain the point $(2, 3, 1)$.
9. $\mathbf{u} = 4\mathbf{i} - \mathbf{k}$ (First line)
10. $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ (Second line)
11. $\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|}$
12. $= \frac{8 - 1}{\sqrt{17} \sqrt{9}}$
13. $= \frac{7}{3\sqrt{17}}$
14. $= \frac{7\sqrt{17}}{51}$