MATH152 CALCULUS II TUTORIAL – IV

(30.10.2015)

Question 1:

Complete the square to write the equation of the sphere in standard form. Find the center and radius.

$9x^2 + 9y^2 + 9z^2 - 6x + 18y + 1 = 0$

1. $9x^2 + 9y^2 + 9z^2 - 6x + 18y + 1 = 0$
2. $x^2 + y^2 + z^2 - \frac{2}{3}x + 2y + \frac{1}{9} = 0$
3. $\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + (y^2 + 2y + 1) + z^2 = -\frac{1}{9} + \frac{1}{9} + 1$
4. $\left(x - \frac{1}{3}\right)^2 + (y + 1)^2 + (z - 0)^2 = 1$
5. Center: $\left(\frac{1}{3}, -1, 0\right)$
6. Radius: 1

Question 2 :

Find the angle θ between the vectors.

$\mathbf{u} = \langle 10, -5, 15\rangle$

$\mathbf{v} = \langle -2, 1, -3\rangle$

1. $\mathbf{u} = -5\mathbf{v} \implies \mathbf{u}$ is parallel to $\mathbf{v}$ and in the opposite direction.
2. $\theta = \pi$

Question 3 :

Let $\mathbf{u} = \langle 3, -2, 1\rangle$, $\mathbf{v} = \langle 2, -4, -3\rangle$, and $\mathbf{w} = \langle -1, 2, 2\rangle$.

Determine the projection of $\mathbf{w}$ onto $\mathbf{u}$.

1. $\text{proj}_{\mathbf{u}}\mathbf{w} = \left(\frac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{u}\|^2}\right)\mathbf{u}$
2. $= -\frac{5}{14}\langle 3, -2, 1\rangle$
3. $= \left\langle -\frac{15}{14}, \frac{10}{14}, -\frac{5}{14}\right\rangle$
4. $= \left\langle -\frac{15}{14}, \frac{5}{7}, -\frac{5}{14}\right\rangle$

Question 4:

Find (a) $\mathbf{u} \times \mathbf{v}$, (b) $\mathbf{v} \times \mathbf{u}$, and (c) $\mathbf{v} \times \mathbf{v}$.

$\mathbf{u} = \langle 7, 3, 2\rangle$

$\mathbf{v} = \langle 1, -1, 5\rangle$

1. (a) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & 2 \\ 1 & -1 & 5 \end{vmatrix}$
2. $= \langle 17, -33, -10\rangle$
3. (b) $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
4. $= \langle -17, 33, 10\rangle$
5. (c) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

Question 5:

Find $\mathbf{u} \times \mathbf{v}$ and show that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

$\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

1. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$
2. $= -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
3. $= \langle -2, 3, -1\rangle$
4. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 1(-2) + 1(3) + 1(-1)$
5. $= 0 \implies \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$
6. $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 2(-2) + 1(3) + (-1)(-1)$
7. $= 0 \implies \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$
8. $(-\mathbf{v}) \times \mathbf{u} = -(\mathbf{v} \times \mathbf{u})$
9. $= \mathbf{u} \times \mathbf{v}$

Question 6:

Verify that the points are the vertices of a parallelogram, and find its area.

(1, 1, 1), (2, 3, 4), (6, 5, 2), (7, 7, 5)

1. $A(1, 1, 1,), B(2, 3, 4), C(6, 5, 2), D(7, 7, 5)$
2. $\overrightarrow{AB} = \langle 1, 2, 3 \rangle$
3. $\overrightarrow{AC} = \langle 5, 4, 1 \rangle$
4. $\overrightarrow{CD} = \langle 1, 2, 3 \rangle$
5. $\overrightarrow{BD} = \langle 5, 4, 1 \rangle$
6. Since $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$, the figure is a parallelogram.
7. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are adjacent sides and
8. $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 5 & 4 & 1 \end{vmatrix}$
9. $= -10\mathbf{i} + 14\mathbf{j} - 6\mathbf{k}.$
10. $A = \|\overrightarrow{AB} \times \overrightarrow{AC}\|$
11. $= \sqrt{332}$
12. $= 2\sqrt{83}$

Question 7:

Find the area of the triangle with the given vertices. (*Hint*: $\frac{1}{2}\|\mathbf{u} \times \mathbf{v}\|$ is the area of the triangle having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.)

(2, −7, 3), (−1, 5, 8), (4, 6, −1)

1. $A(2, -7, 3), B(-1, 5, 8), C(4, 6, -1)$
2. $\overrightarrow{AB} = \langle -3, 12, 5 \rangle$
3. $\overrightarrow{AC} = \langle 2, 13, -4 \rangle$
4. $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 12 & 5 \\ 2 & 13 & -4 \end{vmatrix}$
5. $= \langle -113, -2, -63 \rangle$
6. Area $= \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|$
7. $= \frac{1}{2}\sqrt{16{,}742}$

Question : 8

Find the volume of the parallelepiped shown in Figure 11.42 having $\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ as adjacent edges.

Solution By Theorem 11.10, you have

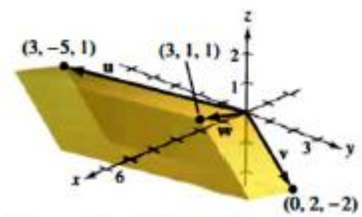
$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \qquad \text{Triple scalar product}$$

$$= \begin{vmatrix} 3 & -5 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} - (-5)\begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} + (1)\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= 3(4) + 5(6) + 1(-6)$$

$$= 36.$$



The parallelepiped has a volume of 36.
Figure 11.42

Question : 9

Find a set of parametric equations of the line.

The line passes through the point (2, 1, 2) and is parallel to the line $x = -t, y = 1 + t, z = -2 + t$.

1. Point: (2, 1, 2)
2. Direction vector: $\langle -1, 1, 1 \rangle$
3. Direction numbers: −1, 1, 1
4. Parametric: $x = 2 - t, y = 1 + t, z = 2 + t$

Question : 10

Determine if any of the lines are parallel or identical.

L_1: $x = 6 - 3t,\ y = -2 + 2t,\ z = 5 + 4t$

L_2: $x = 6t,\ y = 2 - 4t,\ z = 13 - 8t$

L_3: $x = 10 - 6t,\ y = 3 + 4t,\ z = 7 + 8t$

L_4: $x = -4 + 6t,\ y = 3 + 4t,\ z = 5 - 6t$

1. L_1: $\mathbf{v} = \langle -3, 2, 4\rangle$ $(6, -2, 5)$ on line
2. L_2: $\mathbf{v} = \langle 6, -4, -8\rangle$ $(6, -2, 5)$ on line
3. L_3: $\mathbf{v} = \langle -6, 4, 8\rangle$ $(6, -2, 5)$ not on line
4. L_4: $\mathbf{v} = \langle 6, 4, -6\rangle$ not parallel to L_1, L_2, nor L_3
5. Hence, L_1 and L_2 are identical.
6. $L_1 = L_2$ and L_3 are parallel.

Question 11

Determine whether the lines intersect, and if so, find the point of intersection and the cosine of the angle of intersection.

$x = 4t + 2,\ y = 3,\ z = -t + 1$

$x = 2s + 2,\ y = 2s + 3,\ z = s + 1$

1. At the point of intersection, the coordinates for one line equal the corresponding coordinates for the other line. Thus,
2. (i) $4t + 2 = 2s + 2$,
3. (ii) $3 = 2s + 3$,
4. and (iii) $-t + 1 = s + 1$.
5. From (ii), we find that $s = 0$ and consequently,
6. from (iii), $t = 0$.
7. Letting $s = t = 0$, we see that equation (i) is satisfied and therefore the two lines intersect.
8. Substituting zero for s or for t, we obtain the point (2, 3, 1).
9. $\mathbf{u} = 4\mathbf{i} - \mathbf{k}$ (First line)
10. $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ (Second line)
11. $\cos\theta = \dfrac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$
12. $= \dfrac{8 - 1}{\sqrt{17}\sqrt{9}}$
13. $= \dfrac{7}{3\sqrt{17}}$
14. $= \dfrac{7\sqrt{17}}{51}$

Question 12

Equations of lines Let ℓ be the line that passes through the points $P_0(-3, 5, 8)$ and $P_1(4, 2, -1)$.

a. Find an equation of ℓ.

b. Find equations of the projections of ℓ on the xy- and xz-planes.

SOLUTION

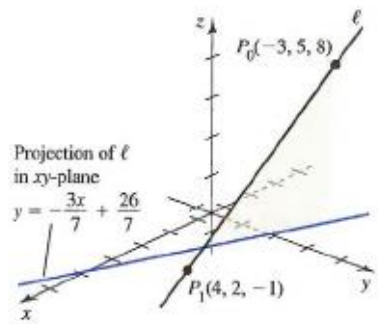
a. The direction of the line is

$$\mathbf{v} = \overrightarrow{P_0P_1} = \langle 4 - (-3), 2 - 5, -1 - 8\rangle = \langle 7, -3, -9\rangle.$$

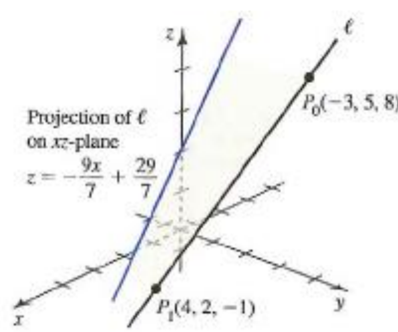
Therefore, with $\mathbf{r}_0 = \langle -3, 5, 8\rangle$, the equation of ℓ is

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t\mathbf{v}\\ &= \langle -3, 5, 8\rangle + t\langle 7, -3, -9\rangle\\ &= \langle -3 + 7t, 5 - 3t, 8 - 9t\rangle.\end{aligned}$$

b. Setting the z-component of the equation of ℓ equal to zero, the parametric equations of the projection of ℓ on the xy-plane are $x = -3 + 7t$, $y = 5 - 3t$. Eliminating t from these equations gives the equation $y = -\frac{3}{7}x + \frac{26}{7}$ (Figure 11.69a). The projection of ℓ on the xz-plane (setting $y = 0$) is $x = -3 + 7t$, $z = 8 - 9t$. Eliminating t gives the equation $z = -\frac{9}{7}x + \frac{29}{7}$ (Figure 11.69b).



(a)



(b)

FIGURE 11.69