MATH152 CALCULUS II   TUTORIAL – IV

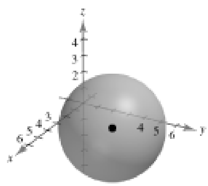
(17.03.2017)

Question 1:

Complete the square to write the equation of the sphere in standard form. Find the center and radius.

$x^2 + y^2 + z^2 - 4x - 6y + 4 = 0$

1. $(x^2 - 4x + 4) + (y^2 - 6y + 9) + z^2 = -4 + 4 + 9$
2. $(x - 2)^2 + (y - 3)^2 + z^2 = 9$
3. Center: $(2, 3, 0)$
4. Radius: 3
5.



Question 2:

Find (a) $\mathbf{u} \cdot \mathbf{v}$, (b) $\mathbf{u} \cdot \mathbf{u}$, (c) $\|\mathbf{u}\|^2$, (d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v}$, and (e) $\mathbf{u} \cdot (2\mathbf{v})$.

$\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \quad \mathbf{v} = \mathbf{i} - \mathbf{k}$

1. (a) $\mathbf{u} \cdot \mathbf{v} = 2(1) + (-1)(0) + 1(-1)$
2. $\quad = 1$
3. (b) $\mathbf{u} \cdot \mathbf{u} = 2(2) + (-1)(-1) + (1)(1)$
4. $\quad = 6$
5. (c) $\|\mathbf{u}\|^2 = 6$
6. (d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = \mathbf{v}$
7. $\quad = \mathbf{i} - \mathbf{k}$
8. (e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) =$
9. $\quad = 2$

Question 3:

Find the angle θ between the vectors.

$\mathbf{u} = \langle 1, 1, 1 \rangle, \mathbf{v} = \langle 2, 1, -1 \rangle$

1. $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$
2. $\quad = \dfrac{2}{\sqrt{3}\sqrt{6}}$
3. $\quad = \dfrac{\sqrt{2}}{3}$
4. $\theta = \arccos \dfrac{\sqrt{2}}{3}$
5. $\quad \approx 61.9°$

Question 4:

Find the angle θ between the vectors.

$\mathbf{u} = \langle 10, -5, 15 \rangle$

$\mathbf{v} = \langle -2, 1, -3 \rangle$

1. $\mathbf{u} = -5\mathbf{v} \implies \mathbf{u}$ is parallel to $\mathbf{v}$ and in the opposite direction.
2. $\theta = \pi$

Question 5:

Determine whether $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, parallel, or neither.

$\mathbf{u} = \langle 4, 3 \rangle, \quad \mathbf{v} = \left\langle \frac{1}{2}, -\frac{2}{3} \right\rangle$

1. $\mathbf{u} \neq c\mathbf{v} \implies$ not parallel
2. $\mathbf{u} \cdot \mathbf{v} = 0 \implies$ orthogonal

Question 6 :

(a) Find the projection of **u** onto **v**, and (b) find the vector component of **u** orthogonal to **v**.

$\mathbf{u} = \langle 2, 1, 2\rangle, \ \mathbf{v} = \langle 0, 3, 4\rangle$

1. (a) $\mathbf{w}_1 = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$
2. $= \dfrac{11}{25}\langle 0, 3, 4\rangle$
3. $= \left\langle 0, \dfrac{33}{25}, \dfrac{44}{25}\right\rangle$
4. (b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$
5. $= \left\langle 2, -\dfrac{8}{25}, \dfrac{6}{25}\right\rangle$

Question 7 :

Let $\mathbf{u} = \langle 3, -2, 1\rangle$, $\mathbf{v} = \langle 2, -4, -3\rangle$, and $\mathbf{w} = \langle -1, 2, 2\rangle$.

Determine the projection of **w** onto **u**.

1. $\text{proj}_{\mathbf{u}}\mathbf{w} = \left(\dfrac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{u}\|^2}\right)\mathbf{u}$
2. $= -\dfrac{5}{14}\langle 3, -2, 1\rangle$
3. $= \left\langle -\dfrac{15}{14}, \dfrac{10}{14}, -\dfrac{5}{14}\right\rangle$
4. $= \left\langle -\dfrac{15}{14}, \dfrac{5}{7}, -\dfrac{5}{14}\right\rangle$

Question 8:

Find (a) $\mathbf{u}\times\mathbf{v}$, (b) $\mathbf{v}\times\mathbf{u}$, and (c) $\mathbf{v}\times\mathbf{v}$.

$\mathbf{u} = \langle 7, 3, 2\rangle$

$\mathbf{v} = \langle 1, -1, 5\rangle$

1. (a) $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & 2 \\ 1 & -1 & 5 \end{vmatrix}$
2. $= \langle 17, -33, -10\rangle$
3. (b) $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v})$
4. $= \langle -17, 33, 10\rangle$
5. (c) $\mathbf{v}\times\mathbf{v} = \mathbf{0}$

Question 9:

Find $\mathbf{u}\times\mathbf{v}$ and show that it is orthogonal to both **u** and **v**.

$\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

1. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$
2. $= -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
3. $= \langle -2, 3, -1\rangle$
4. $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = 1(-2) + 1(3) + 1(-1)$
5. $= 0 \implies \mathbf{u}\perp\mathbf{u}\times\mathbf{v}$
6. $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) = 2(-2) + 1(3) + (-1)(-1)$
7. $= 0 \implies \mathbf{v}\perp\mathbf{u}\times\mathbf{v}$
8. $(-\mathbf{v})\times\mathbf{u} = -(\mathbf{v}\times\mathbf{u})$
9. $= \mathbf{u}\times\mathbf{v}$

Question 10:

Find the area of the triangle with the given vertices. (*Hint*: $\frac{1}{2}\|\mathbf{u}\times\mathbf{v}\|$ is the area of the triangle having **u** and **v** as adjacent sides.)

$(2, -7, 3), (-1, 5, 8), (4, 6, -1)$

1. $A(2, -7, 3), B(-1, 5, 8), C(4, 6, -1)$
2. $\overrightarrow{AB} = \langle -3, 12, 5\rangle$
3. $\overrightarrow{AC} = \langle 2, 13, -4\rangle$
4. $\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 12 & 5 \\ 2 & 13 & -4 \end{vmatrix}$
5. $= \langle -113, -2, -63\rangle$
6. $\text{Area} = \dfrac{1}{2}\|\overrightarrow{AB}\times\overrightarrow{AC}\|$
7. $= \dfrac{1}{2}\sqrt{16{,}742}$

Question : 11

Find the volume of the parallelepiped shown in Figure 11.42 havi
$\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ as adjacent edges.

Solution By Theorem 11.10, you have

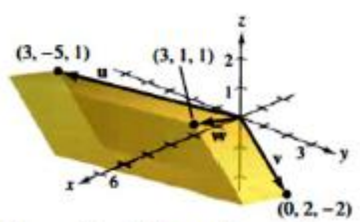
$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \qquad \text{Triple scalar product}$$

$$= \begin{vmatrix} 3 & -5 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} - (-5)\begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} + (1)\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= 3(4) + 5(6) + 1(-6)$$

$$= 36.$$



The parallelepiped has a volume of 36.
Figure 11.42

Question : 12

Find a set of parametric equations of the line.

The line passes through the point $(2, 1, 2)$ and is parallel to the line $x = -t, y = 1 + t, z = -2 + t$.

1. Point: $(2, 1, 2)$
2. Direction vector: $\langle -1, 1, 1\rangle$
3. Direction numbers: $-1, 1, 1$
4. Parametric: $x = 2 - t, y = 1 + t, z = 2 + t$

Question : 13

Determine if any of the lines are parallel or identical.

L_1: $x = 6 - 3t,\ y = -2 + 2t,\ z = 5 + 4t$

L_2: $x = 6t,\ y = 2 - 4t,\ z = 13 - 8t$

L_3: $x = 10 - 6t,\ y = 3 + 4t,\ z = 7 + 8t$

L_4: $x = -4 + 6t,\ y = 3 + 4t,\ z = 5 - 6t$

1. L_1: $\mathbf{v} = \langle -3, 2, 4\rangle$ $\quad (6, -2, 5)$ on line
2. L_2: $\mathbf{v} = \langle 6, -4, -8\rangle$ $\quad (6, -2, 5)$ on line
3. L_3: $\mathbf{v} = \langle -6, 4, 8\rangle$ $\quad (6, -2, 5)$ not on line
4. L_4: $\mathbf{v} = \langle 6, 4, -6\rangle$ $\quad$ not parallel to L_1, L_2, nor L_3
5. Hence, L_1 and L_2 are identical.
6. $L_1 = L_2$ and L_3 are parallel.

Question 14

Determine whether the lines intersect, and if so, find the point of intersection and the cosine of the angle of intersection.

$x = 4t + 2,\ y = 3,\ z = -t + 1$

$x = 2s + 2,\ y = 2s + 3,\ z = s + 1$

1. At the point of intersection, the coordinates for one line equal the corresponding coordinates for the other line. Thus,
2. (i) $4t + 2 = 2s + 2$,
3. (ii) $3 = 2s + 3$,
4. and (iii) $-t + 1 = s + 1$.
5. From (ii), we find that $s = 0$ and consequently,
6. from (iii), $t = 0$.
7. Letting $s = t = 0$, we see that equation (i) is satisfied and therefore the two lines intersect.
8. Substituting zero for s or for t, we obtain the point $(2, 3, 1)$.
9. $\mathbf{u} = 4\mathbf{i} - \mathbf{k}$ $\quad$ (First line)
10. $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ $\quad$ (Second line)
11. $\cos\theta = \dfrac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$
12. $= \dfrac{8 - 1}{\sqrt{17}\sqrt{9}}$
13. $= \dfrac{7}{3\sqrt{17}}$
14. $= \dfrac{7\sqrt{17}}{51}$