MATH152 CALCULUS II   TUTORIAL – 3

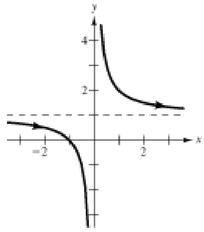
(11.03.2016)

Question 1 :

Sketch the curve represented by the parametric equations (indicate the orientation of the curve), and write the corresponding rectangular equation by eliminating the parameter.

$x = t - 1, \; y = \dfrac{t}{t-1}$

1.



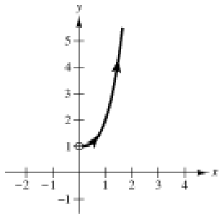
2. $y = \dfrac{x+1}{x}$

Question 2:

Sketch the curve represented by the parametric equations (indicate the orientation of the curve), and write the corresponding rectangular equation by eliminating the parameter.

$x = e^{t}, \; y = e^{3t} + 1$

1.



2. $y = x^3 + 1$
3. $x > 0$

Question 3:

Find two different sets of parametric equations for the rectangular equation.

$y = x^3$

1. Example
2. $x = t, \qquad y = t^3$
3. $x = \sqrt[3]{t}, \qquad y = t$
4. $x = \tan t, \qquad y = \tan^3 t$

Question 4:

Find two different sets of parametric equations for the rectangular equation.

$y = 3x - 2$

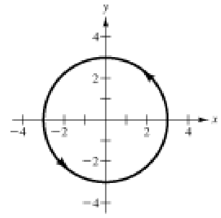
1. Example
2. $x = t, \qquad y = 3t - 2$
3. $x = t - 3, \qquad y = 3t - 11$

Question 5:

Sketch the curve represented by the parametric equations (indicate the orientation of the curve), and write the corresponding rectangular equation by eliminating the parameter.

$x = 3\cos\theta, \; y = 3\sin\theta$

1.



2. Squaring both equations and adding, we have
3. $x^2 + y^2 = 9.$

Question : 6

Find dy/dx and d^2y/dx^2, and find the slope and concavity (if possible) at the given value of the parameter.

Parametric Equations	*Point*
$x = 2\cos\theta,\ y = 2\sin\theta$	$\theta = \dfrac{\pi}{4}$

1. $\dfrac{dy}{dx} = \dfrac{2\cos\theta}{-2\sin\theta}$
2. $= -\cot\theta$
3. $= -1$ when $\theta = \dfrac{\pi}{4}$.
4. $\dfrac{d^2y}{dx^2} = \dfrac{\csc^2\theta}{-2\sin\theta}$
5. $= \dfrac{-\csc^3\theta}{2}$
6. $= -\sqrt{2}$ when $\theta = \dfrac{\pi}{4}$.
7. Concave downward

Question : 7

Without eliminating the parameter, find dy/dx and d^2y/dx^2 at $(1, 1)$ and $(1, -1)$ on the semicubical parabola given by the parametric equations

$$x = t^2, \quad y = t^3 \quad (-\infty < t < +\infty)$$

Solution

From (4) we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3}{2}t \quad (t \neq 0) \tag{7}$$

and from (4) applied to $y' = dy/dx$ we have

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{3/2}{2t} = \frac{3}{4t} \tag{8}$$

Since the point $(1, 1)$ on the curve corresponds to $t = 1$ in the parametric equations, it follows from (7) and (8) that

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{3}{2} \quad \text{and} \quad \left.\frac{d^2y}{dx^2}\right|_{t=1} = \frac{3}{4}$$

Question : 8

Write an integral that represents the arc length of the curve on the given interval. Do not evaluate the integral.

Parametric Equations	*Interval*
$x = 2t - t^2,\ y = 2t^{3/2}$	$1 \le t \le 2$

1. $\dfrac{dx}{dt} = 2 - 2t$
2. $\dfrac{dy}{dt} = 3t^{1/2}$
3. $s = \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
4. $= \displaystyle\int_1^2 \sqrt{(2 - 2t)^2 + (3t^{1/2})^2}\, dt$
5. $= \displaystyle\int_1^2 \sqrt{4 - 8t + 4t^2 + 9t}\, dt$
6. $= \displaystyle\int_1^2 \sqrt{4t^2 + t + 4}\, dt$

Question : 9

Find the arc length of the curve on the given interval.

Parametric Equations	*Interval*
$x = e^{-t}\cos t,\ y = e^{-t}\sin t$	$0 \le t \le \dfrac{\pi}{2}$

1. $\dfrac{dx}{dt} = -e^{-t}(\sin t + \cos t)$
2. $\dfrac{dy}{dt} = e^{-t}(\cos t - \sin t)$
3. $s = \displaystyle\int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
4. $= \displaystyle\int_0^{\pi/2} \sqrt{2e^{-2t}}\, dt$
5. $= -\sqrt{2}\displaystyle\int_0^{\pi/2} e^{-t}(-1)\, dt$
6. $= \left[-\sqrt{2}e^{-t}\right]_0^{\pi/2}$
7. $= \sqrt{2}(1 - e^{-\pi/2})$
8. ≈ 1.12

Question : 10

Find (a) $\frac{2}{3}\mathbf{u}$, (b) $\mathbf{v} - \mathbf{u}$, and (c) $2\mathbf{u} + 5\mathbf{v}$.

$\mathbf{u} = \langle 4, 9\rangle,\ \mathbf{v} = \langle 2, -5\rangle$

1. (a) $\frac{2}{3}\mathbf{u} = \frac{2}{3}\langle 4, 9\rangle$
2. $= \left\langle \frac{8}{3}, 6\right\rangle$
3. (b) $\mathbf{v} - \mathbf{u} = \langle 2, -5\rangle - \langle 4, 9\rangle$
4. $= \langle -2, -14\rangle$
5. (c) $2\mathbf{u} + 5\mathbf{v} = 2\langle 4, 9\rangle + 5\langle 2, -5\rangle$
6. $= \langle 18, -7\rangle$

Question 11

Find the following.

(a) $\|\mathbf{u}\|$ (b) $\|\mathbf{v}\|$ (c) $\|\mathbf{u} + \mathbf{v}\|$

(d) $\left\|\frac{\mathbf{u}}{\|\mathbf{u}\|}\right\|$ (e) $\left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\|$ (f) $\left\|\frac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|}\right\|$

$\mathbf{u} = \left\langle 1, \frac{1}{2}\right\rangle,\ \mathbf{v} = \langle 2, 3\rangle$

1. (a) $\|\mathbf{u}\| = \sqrt{1 + \frac{1}{4}}$
2. $= \frac{\sqrt{5}}{2}$
3. (b) $\|\mathbf{v}\| = \sqrt{4 + 9}$
4. $= \sqrt{13}$
5. (c) $\mathbf{u} + \mathbf{v} = \left\langle 3, \frac{7}{2}\right\rangle$
6. $\|\mathbf{u} + \mathbf{v}\| = \sqrt{9 + \frac{49}{4}}$
7. $= \frac{\sqrt{85}}{2}$
8. (d) $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{2}{\sqrt{5}}\left\langle 1, \frac{1}{2}\right\rangle$
9. $\left\|\frac{\mathbf{u}}{\|\mathbf{u}\|}\right\| = 1$
10. (e) $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{13}}\langle 2, 3\rangle$
11. $\left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = 1$
12. (f) $\frac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|} = \frac{2}{\sqrt{85}}\left\langle 3, \frac{7}{2}\right\rangle$
13. $\left\|\frac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|}\right\| = 1$

Question 12

Find the lengths of the sides of the triangle with the indicated vertices, and determine whether the triangle is a right triangle, an isosceles triangle, or neither.

$(1, -3, -2), (5, -1, 2), (-1, 1, 2)$

1. $A(1, -3, -2), B(5, -1, 2), C(-1, 1, 2)$
2. $|AB| = \sqrt{16 + 4 + 16}$
3. $= 6$
4. $|AC| = \sqrt{4 + 16 + 16}$
5. $= 6$
6. $|BC| = \sqrt{36 + 4 + 0}$
7. $= 2\sqrt{10}$
8. Since $|AB| = |AC|$, the triangle is isosceles.

Question 13

Find a unit vector (a) in the direction of $\mathbf{u}$ and (b) in the direction opposite of $\mathbf{u}$.

$\mathbf{u} = \langle 3, 2, -5\rangle$

1. $\|\mathbf{u}\| = \sqrt{9 + 4 + 25}$
2. $= \sqrt{38}$
3. (a) $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{38}}\langle 3, 2, -5\rangle$
4. (b) $-\frac{\mathbf{u}}{\|\mathbf{u}\|} = -\frac{1}{\sqrt{38}}\langle 3, 2, -5\rangle$

Question 14

Find (a) $\frac{2}{3}\mathbf{u}$, (b) $\mathbf{v} - \mathbf{u}$, and (c) $2\mathbf{u} + 5\mathbf{v}$.

$\mathbf{u} = \langle 4, 9\rangle,\ \mathbf{v} = \langle 2, -5\rangle$

1. (a) $\frac{2}{3}\mathbf{u} = \frac{2}{3}\langle 4, 9\rangle$
2. $= \left\langle \frac{8}{3}, 6\right\rangle$
3. (b) $\mathbf{v} - \mathbf{u} = \langle 2, -5\rangle - \langle 4, 9\rangle$
4. $= \langle -2, -14\rangle$
5. (c) $2\mathbf{u} + 5\mathbf{v} = 2\langle 4, 9\rangle + 5\langle 2, -5\rangle$
6. $= \langle 18, -7\rangle$