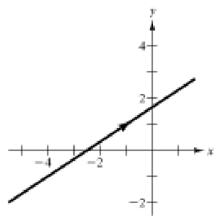
MATH152 CALCULUS II TUTORIAL – II

(27.03.2015)

Question 1 :

Sketch the curve represented by the parametric equations (indicate the orientation of the curve), and write the corresponding rectangular equation by eliminating the parameter.

$x = 3t - 1,\ y = 2t + 1$

1. 

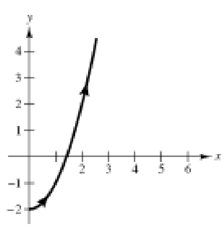
2. $y = 2\left(\frac{x+1}{3}\right) + 1$

3. $2x - 3y + 5 = 0$

Question 2:

Sketch the curve represented by the parametric equations (indicate the orientation of the curve), and write the corresponding rectangular equation by eliminating the parameter.

$x = \sqrt{t},\ y = t - 2$

1. 

2. $y = x^2 - 2$

3. $x \geq 0$

Question 3:

Find two different sets of parametric equations for the rectangular equation.

$y = x^3$

1. Example
2. $x = t, \qquad y = t^3$
3. $x = \sqrt[3]{t}, \qquad y = t$
4. $x = \tan t, \qquad y = \tan^3 t$

Question 4:

Classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

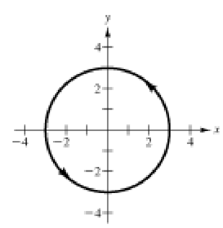
$x^2 + 4y^2 - 6x + 16y + 21 = 0$

1. $(x^2 - 6x + 9) + 4(y^2 + 4y + 4) = -21 + 9 + 16$
2. $(x - 3)^2 + 4(y + 2)^2 = 4$
3. Ellipse

Question 5:

Sketch the curve represented by the parametric equations (indicate the orientation of the curve), and write the corresponding rectangular equation by eliminating the parameter.

$x = 3 \cos \theta,\ y = 3 \sin \theta$

1. 

2. Squaring both equations and adding, we have
3. $x^2 + y^2 = 9.$

Question : 6

Find the slope of the tangent line to the unit circle

$$x = \cos t, \quad y = \sin t \qquad (0 \le t \le 2\pi)$$

at the point where $t = \pi/6$ (Figure 10.1.9).

Solution

From (4), the slope at a general point on the circle is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t$$

Thus, the slope at $t = \pi/6$ is

$$\left.\frac{dy}{dx}\right|_{t=\pi/6} = -\cot\frac{\pi}{6} = -\sqrt{3}$$ ◀

Question : 7

Without eliminating the parameter, find dy/dx and d^2y/dx^2 at $(1, 1)$ and $(1, -1)$ on the semicubical parabola given by the parametric equations

$$x = t^2, \quad y = t^3 \qquad (-\infty < t < +\infty)$$

Solution

From (4) we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3}{2}t \qquad (t \ne 0) \tag{7}$$

and from (4) applied to $y' = dy/dx$ we have

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{3/2}{2t} = \frac{3}{4t} \tag{8}$$

Since the point $(1, 1)$ on the curve corresponds to $t = 1$ in the parametric equations, i follows from (7) and (8) that

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{3}{2} \quad \text{and} \quad \left.\frac{d^2y}{dx^2}\right|_{t=1} = \frac{3}{4}$$

Question : 8

Write an integral that represents the arc length of the curve on the given interval. Do not evaluate the integral.

Parametric Equations	*Interval*
$x = 2t - t^2,\ y = 2t^{3/2}$	$1 \le t \le 2$

1. $\frac{dx}{dt} = 2 - 2t$
2. $\frac{dy}{dt} = 3t^{1/2}$
3. $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
4. $= \int_1^2 \sqrt{(2 - 2t)^2 + (3t^{1/2})^2}\, dt$
5. $= \int_1^2 \sqrt{4 - 8t + 4t^2 + 9t}\, dt$
6. $= \int_1^2 \sqrt{4t^2 + t + 4}\, dt$

Question : 9

Find the lengths of the sides of the triangle with the indicated vertices, and determine whether the triangle is a right triangle, an isosceles triangle, or neither.

$(0, 0, 0), (2, 2, 1), (2, -4, 4)$

1. $A(0, 0, 0), B(2, 2, 1), C(2, -4, 4)$
2. $|AB| = \sqrt{4 + 4 + 1}$
3. $= 3$
4. $|AC| = \sqrt{4 + 16 + 16}$
5. $= 6$
6. $|BC| = \sqrt{0 + 36 + 9}$
7. $= 3\sqrt{5}$
8. $|BC|^2 = |AB|^2 + |AC|^2$
9. Right triangle

Question : 10

Find a and b such that $\mathbf{v} = a\mathbf{u} + b\mathbf{w}$, where $\mathbf{u} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 1, -1 \rangle$.

$\mathbf{v} = \langle 3, 0 \rangle$

1. $\mathbf{v} = 3\mathbf{i}$
2. Therefore, $a + b = 3$,
3. $2a - b = 0$.
4. Solving simultaneously, we have $a = 1$,
5. $b = 2$.

Question 11

Determine whether $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, parallel, or neither.

$\mathbf{u} = \langle 2, -3, 1 \rangle$, $\mathbf{v} = \langle -1, -1, -1 \rangle$

1. $\mathbf{u} \neq c\mathbf{v} \implies$ not parallel
2. $\mathbf{u} \cdot \mathbf{v} = 0 \implies$ orthogonal

Question 12

Find the lengths of the sides of the triangle with the indicated vertices, and determine whether the triangle is a right triangle, an isosceles triangle, or neither.

$(1, -3, -2), (5, -1, 2), (-1, 1, 2)$

1. $A(1, -3, -2), B(5, -1, 2), C(-1, 1, 2)$
2. $|AB| = \sqrt{16 + 4 + 16}$
3. $= 6$
4. $|AC| = \sqrt{4 + 16 + 16}$
5. $= 6$
6. $|BC| = \sqrt{36 + 4 + 0}$
7. $= 2\sqrt{10}$
8. Since $|AB| = |AC|$, the triangle is isosceles.

Question 13

Find a unit vector (a) in the direction of $\mathbf{u}$ and (b) in the direction opposite of $\mathbf{u}$.

$\mathbf{u} = \langle 3, 2, -5 \rangle$

1. $\|\mathbf{u}\| = \sqrt{9 + 4 + 25}$
2. $= \sqrt{38}$
3. (a) $\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \dfrac{1}{\sqrt{38}}\langle 3, 2, -5 \rangle$
4. (b) $-\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = -\dfrac{1}{\sqrt{38}}\langle 3, 2, -5 \rangle$

Question 14

Find (a) $\frac{2}{3}\mathbf{u}$, (b) $\mathbf{v} - \mathbf{u}$, and (c) $2\mathbf{u} + 5\mathbf{v}$.

$\mathbf{u} = \langle 4, 9 \rangle$, $\mathbf{v} = \langle 2, -5 \rangle$

1. (a) $\frac{2}{3}\mathbf{u} = \frac{2}{3}\langle 4, 9 \rangle$
2. $= \left\langle \frac{8}{3}, 6 \right\rangle$
3. (b) $\mathbf{v} - \mathbf{u} = \langle 2, -5 \rangle - \langle 4, 9 \rangle$
4. $= \langle -2, -14 \rangle$
5. (c) $2\mathbf{u} + 5\mathbf{v} = 2\langle 4, 9 \rangle + 5\langle 2, -5 \rangle$
6. $= \langle 18, -7 \rangle$

Question 15

Find (a) $\mathbf{u} \cdot \mathbf{v}$, (b) $\mathbf{u} \cdot \mathbf{u}$, (c) $\|\mathbf{u}\|^2$, (d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v}$, and (e) $\mathbf{u} \cdot (2\mathbf{v})$.

$\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{k}$

1. (a) $\mathbf{u} \cdot \mathbf{v} = 2(1) + (-1)(0) + 1(-1)$
2. $= 1$
3. (b) $\mathbf{u} \cdot \mathbf{u} = 2(2) + (-1)(-1) + (1)(1)$
4. $= 6$
5. (c) $\|\mathbf{u}\|^2 = 6$
6. (d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = \mathbf{v}$
7. $= \mathbf{i} - \mathbf{k}$
8. (e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) =$
9. $= 2$

Question : 16 Find

Find $\text{proj}_{\mathbf{v}}\mathbf{u}$ and $\text{scal}_{\mathbf{v}}\mathbf{u}$ for

a. $\mathbf{u} = \langle 4, 1\rangle, \mathbf{v} = \langle 3, 4\rangle$ **b.** $\mathbf{u} = \langle -4, -3\rangle, \mathbf{v} = \langle 1, -1\rangle$

SOLUTION

a. The scalar component of **u** in the direction of **v**

$$\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 4,1\rangle\cdot\langle 3,4\rangle}{|\langle 3,4\rangle|} = \frac{16}{5}.$$

Because $\frac{\mathbf{v}}{|\mathbf{v}|} = \langle \frac{3}{5}, \frac{4}{5}\rangle$, we have

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \text{scal}_{\mathbf{v}}\mathbf{u}\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \frac{16}{5}\left\langle \frac{3}{5}, \frac{4}{5}\right\rangle = \frac{16}{25}\langle 3, 4\rangle$$

b. Using another formula for $\text{proj}_{\mathbf{v}}\mathbf{u}$, we have

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{\langle -4,-3\rangle\cdot\langle 1,-1\rangle}{\langle 1,-1\rangle\cdot\langle 1,-1\rangle}\right)\langle 1,-1\rangle = -\frac{1}{2}\langle 1,$$

Question : 17

Find the following.

(a) $\|\mathbf{u}\|$ (b) $\|\mathbf{v}\|$ (c) $\|\mathbf{u}+\mathbf{v}\|$

(d) $\left\|\frac{\mathbf{u}}{\|\mathbf{u}\|}\right\|$ (e) $\left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\|$ (f) $\left\|\frac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|}\right\|$

$\mathbf{u} = \langle 1, -1\rangle,\ \mathbf{v} = \langle -1, 2\rangle$

1. (a) $\|\mathbf{u}\| = \sqrt{1+1}$
2. $= \sqrt{2}$
3. (b) $\|\mathbf{v}\| = \sqrt{1+4}$
4. $= \sqrt{5}$
5. (c) $\mathbf{u}+\mathbf{v} = \langle 0, 1\rangle$
6. $\|\mathbf{u}+\mathbf{v}\| = \sqrt{0+1}$
7. $= 1$
8. (d) $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{2}}\langle 1, -1\rangle$
9. $\left\|\frac{\mathbf{u}}{\|\mathbf{u}\|}\right\| = 1$
10. (e) $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{5}}\langle -1, 2\rangle$
11. $\left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = 1$
12. (f) $\frac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|} = \langle 0, 1\rangle$
13. $\left\|\frac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|}\right\| = 1$