MATH152 CALCULUS II TUTORIAL – II

(01.03.2019)

Question 1 :

Find the values of x for which the series converges.

$$\sum_{n=0}^{\infty} 2\left(\frac{x}{3}\right)^n$$

1. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2(x/3)^{n+1}}{2(x/3)^n}\right|$
2. $= \lim_{n\to\infty}\left|\frac{x}{3}\right|$
3. $= \left|\frac{x}{3}\right|$
4. For the series to converge: $\left|\frac{x}{3}\right| < 1$
5. $\Longrightarrow -3 < x < 3.$
6. For $x = 3$, the series diverges.
7. For $x = -3$, the series diverges.
8. Answer: $-3 < x < 3$

Question 2 :

Find the values of x for which the series converges.

$$\sum_{n=0}^{\infty} n!\left(\frac{x}{2}\right)^n$$

1. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!|x/2|^{n+1}}{n!|x/2|^n}$
2. $= \lim_{n\to\infty}(n+1)\left|\frac{x}{2}\right|$
3. $= \infty$
4. The series converges only at $x = 0$.

Question 3 :

Find the values of x for which the series converges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{n}$$

1. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}/(n+1)}{x^n/n}\right|$
2. $= \lim_{n\to\infty}\left|\frac{n}{n+1}(x+1)\right|$
3. $= |x+1|$
4. For the series to converge,

   $|x+1| < 1$
5. $\Longrightarrow -1 < x+1 < 1$
6. $\Longrightarrow -2 < x < 0.$
7. For $x = 0$, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$,
8. converges.
9. For $x = -2$, $\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n}$
10. $= \sum_{n=1}^{\infty}\frac{1}{n}$,
11. diverges.
12. Answer: $-2 < x \le 0$

Question 4 :

Find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=0}^{\infty} n!(x-2)^n$$

1. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x-2)^{n+1}}{n!(x-2)^n}\right|$
2. $= \infty$
3. which implies that the series converges only at the center $x = 2$.

Question 5 :

a) Find the Maclaurin series of $f(x)=\ln(1+x^2)$ in powers of x.

b) Find the Taylor series of $f(x)=\dfrac{1}{x}$ in powers of $x-1$.

$$a)\ \frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n \left(\text{converges for } |x|<1\right)\Rightarrow\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}\Rightarrow\frac{2x}{1+x^2}=2\sum_{n=0}^{\infty}(-1)^n x^{2n+1}$$

$$\Rightarrow\int\frac{2x}{1+x^2}dx=2\sum_{n=0}^{\infty}(-1)^n\int x^{2n+1}dx\Rightarrow\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{n+1}\Rightarrow x^2\sum_{n=0}^{\infty}(-1)^n x^{2n}\Rightarrow$$

$$\ln\left(1+x^2\right)+c=x^2\sum_{n=0}^{\infty}(-1)^n x^{2n}\Rightarrow\text{take } x=0\Rightarrow\ln 1+c=0\Rightarrow c=0$$

$$\Rightarrow\ln\left(1+x^2\right)=x^2\sum_{n=0}^{\infty}(-1)^n x^{2n},\ \ -1<x<1,$$

$$b)\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n\Rightarrow\frac{1}{x}=\frac{1}{1+(x-1)}=\sum_{n=0}^{\infty}(-1)^n(x-1)^n\Rightarrow\left(\text{converges for } |x-1|<1\right)$$

Question 6:

a) Find the Maclaurin series of $f(x)=\ln(1+x^2)$ in powers of x.

b) Find the Taylor series of $f(x)=\dfrac{1}{x}$ in powers of $x-1$.

$$a)\ \frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n \left(\text{converges for } |x|<1\right)\Rightarrow\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}\Rightarrow\frac{2x}{1+x^2}=2\sum_{n=0}^{\infty}(-1)^n x^{2n+1}$$

$$\Rightarrow\int\frac{2x}{1+x^2}dx=2\sum_{n=0}^{\infty}(-1)^n\int x^{2n+1}dx\Rightarrow\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{n+1}\Rightarrow x^2\sum_{n=0}^{\infty}(-1)^n x^{2n}\Rightarrow$$

$$\ln\left(1+x^2\right)+c=x^2\sum_{n=0}^{\infty}(-1)^n x^{2n}\Rightarrow\text{take } x=0\Rightarrow\ln 1+c=0\Rightarrow c=0$$

$$\Rightarrow\ln\left(1+x^2\right)=x^2\sum_{n=0}^{\infty}(-1)^n x^{2n},\ \ -1<x<1,$$

$$b)\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n\Rightarrow\frac{1}{x}=\frac{1}{1+(x-1)}=\sum_{n=0}^{\infty}(-1)^n(x-1)^n\Rightarrow\left(\text{converges for } |x-1|<1\right)$$

Question 7:

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$f(x) = -\frac{1}{(x+1)^2} = \frac{d}{dx}\left[\frac{1}{x+1}\right]$$

1. By taking the first derivative,

   we have $\frac{d}{dx}\left[\frac{1}{x+1}\right] = \frac{-1}{(x+1)^2}$.

2. Therefore,

   $$\frac{-1}{(x+1)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty} (-1)^n x^n\right]$$

3. $$= \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$$

4. $$= \sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n,$$

5. $$-1 < x < 1.$$

Question 8:

Find a power series for the function, centered at c, and determine the interval of convergence.

$$f(x) = \frac{2}{1-x^2}, \quad c = 0$$

1. $$\frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$$

2. Writing $f(x)$ as a sum of two geometric series, we have

   $$\frac{2}{1-x^2} = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (-x)^n$$

3. $$= \sum_{n=0}^{\infty} (1 + (-1)^n)x^n$$

4. $$= \sum_{n=0}^{\infty} 2x^{2n}.$$

5. The interval of convergence is $|x^2| < 1$

6. or $-1 < x < 1$ since

7. $$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2x^{2n+2}}{2x^{2n}}\right|$$

8. $$= |x^2|.$$

Question 9:

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$f(x) = \ln(x+1) = \int \frac{1}{1+x}\, dx$$

1. By integrating, we have $\int \frac{1}{x+1}\, dx = \ln(x+1)$.
2. Therefore,

$$\ln(x+1) = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] dx$$

3. $$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1},$$

4. $$-1 < x \leq 1.$$

5. To solve for C, let $x = 0$ and conclude that $C = 0$.
6. Therefore,

$$\ln(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1},$$

7. $$-1 < x \leq 1.$$

Question 10:

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$f(x) = -\frac{1}{(x+1)^2} = \frac{d}{dx}\left[\frac{1}{x+1}\right]$$

1. By taking the first derivative, we have $\frac{d}{dx}\left[\frac{1}{x+1}\right] = \frac{-1}{(x+1)^2}$.
2. Therefore,

$$\frac{-1}{(x+1)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty} (-1)^n x^n\right]$$

3. $$= \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$$

4. $$= \sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n,$$

5. $$-1 < x < 1.$$

Question 11:

Approximating a real number using Taylor polynomials Use polynomials of order $n = 0, 1, 2,$ and 3 to approximate $\sqrt{18}$.

SOLUTION Letting $f(x) = \sqrt{x}$, we choose the center $a = 16$ because it is near 18, and f and its derivatives are easy to evaluate at 16. The Taylor polynomials have the form

$$p_n(x) = f(16) + f'(16)(x-16) + \frac{f''(16)}{2!}(x-16)^2 + \cdots + \frac{f^{(n)}(16)}{n!}(x-16)^n.$$

We now evaluate the required derivatives:

$$f(x) = \sqrt{x} \Rightarrow f(16) = 4$$

$$f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(16) = \frac{1}{8}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \Rightarrow f''(16) = -\frac{1}{256}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} \Rightarrow f'''(16) = \frac{3}{8192}$$

Therefore, the polynomial p_3 (which includes p_0, p_1, and p_2) is

$$p_3(x) = \underbrace{\underbrace{\underbrace{4}_{p_0} + \frac{1}{8}(x-16)}_{p_1} - \frac{1}{512}(x-16)^2}_{p_2} + \frac{1}{16{,}384}(x-16)^3.$$

Table 9.2

n	Approximations $p_n(18)$	Absolute Error $\lvert\sqrt{18} - p_n(18)\rvert$
0	4	2.43×10^{-1}
1	4.25	7.36×10^{-3}
2	4.242188	4.53×10^{-4}
3	4.242676	3.51×10^{-5}

Question : 12

Find a function represented by the series and give the domain of the function.

$$1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots$$

1. $1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots = \sum_{n=0}^{\infty} \left(\frac{2x}{3}\right)^n$

2. $= \frac{1}{1 - (2x/3)}$

3. $= \frac{3}{3 - 2x}$

4. $-\frac{3}{2} < x < \frac{3}{2}$

Question 13

Find the series representation of the function defined by the integral.

$$\int_0^x \frac{\ln(t + 1)}{t}\, dt$$

1. $\frac{1}{1 + t} = \sum_{n=0}^{\infty} (-1)^n t^n$

2. $\ln(1 + t) = \int \frac{1}{1 + t}\, dt$

3. $= \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n + 1}$

4. $\frac{\ln(t + 1)}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n + 1}$

5. $\int_0^x \frac{\ln(t + 1)}{t}\, dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{(n + 1)^2}\right]_0^x$

6. $= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n + 1)^2}$

Question 14

Use the series representation of the function f to find $\lim_{x \to 0} f(x)$ (if it exists).

$$f(x) = \frac{1 - \cos x}{x}$$

1. Since

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

2. $= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$

3. $1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots$

$\cdot \frac{x^7}{8!} + \cdots$

7. we have $\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \sum_{n=0}^{\infty} \frac{(-1)x^{2n+1}}{(2n + 2)!}$

8. $= 0.$