MATH152 CALCULUS II TUTORIAL – II

(23.02.2018)

Question 1 :

Find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=1}^{\infty} \frac{n}{n+1}(-2x)^{n-1}$$

1. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(-2x)^n}{n+2} \cdot \frac{n+1}{n(-2x)^{n-1}}\right|$
2. $= \lim_{n\to\infty} \left|\frac{(-2x)(n+1)^2}{n(n+2)}\right|$
3. $= 2|x|$
4. $R = \frac{1}{2}$
5. Interval: $-\frac{1}{2} < x < \frac{1}{2}$
6. When $x = -\frac{1}{2}$, the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges by the nth Term Test.
7. When $x = \frac{1}{2}$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n+1}$ diverges.
8. Therefore, the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$.

Question 2 :

Find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=0}^{\infty} (2n)!\left(\frac{x}{2}\right)^n$$

1. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(2n+2)!x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(2n)!x^n}\right|$
2. $= \lim_{n\to\infty} \left|\frac{(2n+2)(2n+1)x}{2}\right|$
3. $= \infty$
4. Therefore, the series converges only for $x = 0$.

Question 3 :

Find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$$

1. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+2}(x-5)^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n5^n}{(-1)^{n+1}(x-5)^n}\right|$
2. $= \lim_{n\to\infty} \left|\frac{n(x-5)}{5(n+1)}\right|$
3. $= \frac{1}{5}|x-5|$
4. $R = 5$
5. Center: $x = 5$
6. Interval: $-5 < x - 5 < 5$ or $0 < x < 10$
7. When $x = 0$, the p-series $\sum_{n=1}^{\infty} \frac{-1}{n}$ diverges.
8. When $x = 10$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.
9. Therefore, the interval of convergence is $0 < x \le 10$.

Question 4 :

Find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$$

1. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+2}(x-1)^{n+2}}{n+2} \cdot \frac{n+1}{(-1)^{n+1}(x-1)^{n+1}}\right|$
2. $= \lim_{n\to\infty} \left|\frac{(n+1)(x-1)}{n+2}\right|$
3. $= |x-1|$
4. $R = 1$
5. Center: $x = 1$
6. Interval: $-1 < x - 1 < 1$ or $0 < x < 2$
7. When $x = 0$, the series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges by the integral test.
8. When $x = 2$, the alternating series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$ converges.
9. Therefore, the interval of convergence is $0 < x \le 2$.

Question 5 :

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$h(x) = \frac{-2}{x^2-1} = \frac{1}{1+x} + \frac{1}{1-x}$$

1. $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$
2. $\frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^n (-x)^n$
3. $= \sum_{n=0}^{\infty} (-1)^{2n} x^n$
4. $= \sum_{n=0}^{\infty} x^n$
5. $h(x) = \frac{-2}{x^2-1}$
6. $= \frac{1}{1+x} + \frac{1}{1-x}$
7. $= \sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n$
8. $= \sum_{n=0}^{\infty} [(-1)^n + 1] x^n$
9. $= 2 + 0x + 2x^2 + 0x^3 + 2x^4 + 0x^5 + 2x^6 + \cdots$
10. $= \sum_{n=0}^{\infty} 2x^{2n},$
11. $-1 < x < 1$
12. (See Exercise 15.)

Question 6 :

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$f(x) = -\frac{1}{(x+1)^2} = \frac{d}{dx}\left[\frac{1}{x+1}\right]$$

1. By taking the first derivative, we have $\frac{d}{dx}\left[\frac{1}{x+1}\right] = \frac{-1}{(x+1)^2}.$
2. Therefore, $\frac{-1}{(x+1)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty} (-1)^n x^n\right]$
3. $= \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$
4. $= \sum_{n=0}^{\infty} (-1)^{n+1} (n+1) x^n,$
5. $-1 < x < 1.$

Question 7 :

Find a power series for the function, centered at c, and determine the interval of convergence.

$$g(x) = \frac{3x}{x^2 + x - 2}, \quad c = 0$$

1. $$\frac{3x}{x^2 + x - 2} = \frac{2}{x+2} + \frac{1}{x-1}$$
2. $$= \frac{2}{2+x} + \frac{1}{-1+x}$$
3. $$= \frac{1}{1+(1/2)x} + \frac{-1}{1-x}$$
4. Writing $f(x)$ as a sum of two geometric series, we have
$$\frac{3x}{x^2 + x - 2} = \sum_{n=0}^{\infty}\left(-\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty}(-1)(x)^n$$
5. $$= \sum_{n=0}^{\infty}\left[\frac{1}{(-2)^n} - 1\right]x^n.$$
6. The interval of convergence is $-1 < x < 1$ since
7. $$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(1-(-2)^{n+1})x^{n+1}}{(-2)^{n+1}} \cdot \frac{(-2)^n}{(1-(-2)^n)x^n}\right|$$
8. $$= \lim_{n\to\infty}\left|\frac{(1-(-2)^{n+1})x}{-2-(-2)^{n+1}}\right|$$
9. $$= |x|.$$

Question 8:

Find a power series for the function, centered at c, and determine the interval of convergence.

$$f(x) = \frac{3}{x+2}, \quad c = 0$$

1. Writing $f(x)$ in the form $a/(1-r)$, we have
$$\frac{3}{x+2} = \frac{3}{2+x}$$
2. $$= \frac{3/2}{1+(1/2)x}$$
3. $$= \frac{a}{1-r}$$
4. which implies that $a = 3/2$ and $r = (-1/2)x$.
5. Therefore, the power series for $f(x)$ is given by
$$\frac{3}{x+2} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}\frac{3}{2}\left(-\frac{1}{2}x\right)^n$$
6. $$= 3\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}$$
7. $$= \frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n,$$
8. $|x| < 2$
9. or $-2 < x < 2$.

Question 9:

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$f(x) = \ln(x+1) = \int \frac{1}{1+x}\,dx$$

1. By integrating, we have $\int \frac{1}{x+1}\,dx = \ln(x+1)$.
2. Therefore,
$$\ln(x+1) = \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx$$
3. $$= C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1},$$
4. $$-1 < x \le 1.$$
5. To solve for C, let $x = 0$ and conclude that $C = 0$.
6. Therefore,
$$\ln(x+1) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1},$$
7. $$-1 < x \le 1.$$

Question 10:

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$f(x) = -\frac{1}{(x+1)^2} = \frac{d}{dx}\left[\frac{1}{x+1}\right]$$

1. By taking the first derivative,
we have $\frac{d}{dx}\left[\frac{1}{x+1}\right] = \frac{-1}{(x+1)^2}$.
2. Therefore,
$$\frac{-1}{(x+1)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]$$
3. $$= \sum_{n=1}^{\infty}(-1)^n n x^{n-1}$$
4. $$= \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n,$$
5. $$-1 < x < 1.$$

Question 11:

Approximating a real number using Taylor polynomials Use polynomials of order $n = 0, 1, 2,$ and 3 to approximate $\sqrt{18}$.

SOLUTION Letting $f(x) = \sqrt{x}$, we choose the center $a = 16$ because it is near and its derivatives are easy to evaluate at 16. The Taylor polynomials have the fo

$$p_n(x) = f(16) + f'(16)(x - 16) + \frac{f''(16)}{2!}(x - 16)^2 + \cdots + \frac{f^{(n)}(16)}{n!}(x -$$

We now evaluate the required derivatives:

$$f(x) = \sqrt{x} \Rightarrow f(16) = 4$$

$$f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(16) = \frac{1}{8}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \Rightarrow f''(16) = -\frac{1}{256}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} \Rightarrow f'''(16) = \frac{3}{8192}$$

Therefore, the polynomial p_3 (which includes p_0, p_1, and p_2) is

$$p_3(x) = \underbrace{\underbrace{\underbrace{4}_{p_0} + \frac{1}{8}(x - 16)}_{p_1} - \frac{1}{512}(x - 16)^2}_{p_2} + \frac{1}{16{,}384}(x -$$

Table 9.2

n	Approximations $p_n(18)$	Absolute Error $\lvert\sqrt{18} - p_n(18)\rvert$
0	4	2.43×10^{-1}
1	4.25	7.36×10^{-3}
2	4.242188	4.53×10^{-4}
3	4.242676	3.51×10^{-5}

Question : 12

Find a function represented by the series and give the domain of the function.

$$1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots$$

1. $1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots = \sum_{n=0}^{\infty}\left(\frac{2x}{3}\right)^n$

2. $= \frac{1}{1 - (2x/3)}$

3. $= \frac{3}{3 - 2x}$

4. $-\frac{3}{2} < x < \frac{3}{2}$

Question 13

Find the series representation of the function defined by the integral.

$$\int_0^x \frac{\ln(t + 1)}{t}\,dt$$

1. $\frac{1}{1 + t} = \sum_{n=0}^{\infty}(-1)^n t^n$

2. $\ln(1 + t) = \int \frac{1}{1 + t}\,dt$

3. $= \sum_{n=0}^{\infty}\frac{(-1)^n t^{n+1}}{n + 1}$

4. $\frac{\ln(t + 1)}{t} = \sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n + 1}$

5. $\int_0^x \frac{\ln(t + 1)}{t}\,dt = \left[\sum_{n=0}^{\infty}\frac{(-1)^n t^{n+1}}{(n + 1)^2}\right]_0^x$

6. $= \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{(n + 1)^2}$

Question 14

Use the series representation of the function f to find $\lim_{x\to 0} f(x)$ (if it exists).

$$f(x) = \frac{1 - \cos x}{x}$$

1. Since

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

2. $= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$

3. $1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots$

$\frac{x^7}{8!} + \cdots$

7. we have $\lim_{x\to 0}\frac{1 - \cos x}{x} = \lim_{x\to 0}\sum_{n=0}^{\infty}\frac{(-1)x^{2n+1}}{(2n + 2)!}$

8. $= 0.$