MATH152 CALCULUS II TUTORIAL – II

(06.10.2017)

Question 1 :

Find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$$

1. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{n+1}}{n+1} \cdot \frac{n}{(-1)^n x^n}\right|$
2. $= \lim_{n\to\infty} \left|\frac{nx}{n+1}\right|$
3. $= |x|$
4. Interval: $-1 < x < 1$
5. When $x = 1$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.
6. When $x = -1$, the p-series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
7. Therefore, the interval of convergence is $-1 < x \leq 1$.

Question 2 :

Find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

1. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$
2. $= \lim_{n\to\infty} \left|\frac{x}{n+1}\right|$
3. $= 0$
4. The series converges for all x.
5. Therefore, the interval of convergence is $-\infty < x < \infty$.

Question 3 :

Find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$$

1. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+2}(x-5)^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n5^n}{(-1)^{n+1}(x-5)^n}\right|$
2. $= \lim_{n\to\infty} \left|\frac{n(x-5)}{5(n+1)}\right|$
3. $= \frac{1}{5}|x-5|$
4. $R = 5$
5. Center: $x = 5$
6. Interval: $-5 < x - 5 < 5$ or $0 < x < 10$
7. When $x = 0$, the p-series $\sum_{n=1}^{\infty} \frac{-1}{n}$ diverges.
8. When $x = 10$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.
9. Therefore, the interval of convergence is $0 < x \leq 10$.

Question 4 :

Find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=0}^{\infty} n!(x-2)^n$$

1. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!(x-2)^{n+1}}{n!(x-2)^n}\right|$
2. $= \infty$
3. which implies that the series converges only at the center $x = 2$.

Question 5 :

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$h(x) = \frac{-2}{x^2-1} = \frac{1}{1+x} + \frac{1}{1-x}$$

1. $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$
2. $\frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^n (-x)^n$
3. $= \sum_{n=0}^{\infty} (-1)^{2n} x^n$
4. $= \sum_{n=0}^{\infty} x^n$
5. $h(x) = \frac{-2}{x^2-1}$
6. $= \frac{1}{1+x} + \frac{1}{1-x}$
7. $= \sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n$
8. $= \sum_{n=0}^{\infty} [(-1)^n + 1] x^n$
9. $= 2 + 0x + 2x^2 + 0x^3 + 2x^4 + 0x^5 + 2x^6 + \cdots$
10. $= \sum_{n=0}^{\infty} 2x^{2n}$,
11. $-1 < x < 1$
12. (See Exercise 15.)

Question 6 :

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$g(x) = \frac{1}{x^2+1}$$

1. $\frac{1}{x^2+1} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n$
2. $= \sum_{n=0}^{\infty} (-1)^n x^{2n}$,
3. $-1 < x < 1$

Question 7 :

Find a power series for the function, centered at c, and determine the interval of convergence.

$$g(x) = \frac{3x}{x^2+x-2}, \quad c = 0$$

1. $\frac{3x}{x^2+x-2} = \frac{2}{x+2} + \frac{1}{x-1}$
2. $= \frac{2}{2+x} + \frac{1}{-1+x}$
3. $= \frac{1}{1+(1/2)x} + \frac{-1}{1-x}$
4. Writing $f(x)$ as a sum of two geometric series, we have

   $\frac{3x}{x^2+x-2} = \sum_{n=0}^{\infty} \left(-\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty} (-1)(x)^n$
5. $= \sum_{n=0}^{\infty} \left[\frac{1}{(-2)^n} - 1\right] x^n.$
6. The interval of convergence is $-1 < x < 1$ since
7. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(1-(-2)^{n+1})x^{n+1}}{(-2)^{n+1}} \cdot \frac{(-2)^n}{(1-(-2)^n)x^n}\right|$
8. $= \lim_{n\to\infty} \left|\frac{(1-(-2)^{n+1})x}{-2-(-2)^{n+1}}\right|$
9. $= |x|.$

Question 8:

Find a power series for the function, centered at c, and determine the interval of convergence.

$$f(x) = \frac{3}{x+2}, \quad c = 0$$

1. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{x+2} = \frac{3}{2+x}$$

2. $$= \frac{3/2}{1+(1/2)x}$$

3. $$= \frac{a}{1-r}$$

4. which implies that $a = 3/2$ and $r = (-1/2)x$.
5. Therefore, the power series for $f(x)$ is given by

$$\frac{3}{x+2} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{3}{2}\left(-\frac{1}{2}x\right)^n$$

6. $$= 3\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}}$$

7. $$= \frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n,$$

8. $|x| < 2$
9. or $-2 < x < 2$.

Question 9:

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$f(x) = \ln(x+1) = \int \frac{1}{1+x}\,dx$$

1. By integrating, we have $\displaystyle\int \frac{1}{x+1}\,dx = \ln(x+1)$.
2. Therefore,

$$\ln(x+1) = \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx$$

3. $$= C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1},$$

4. $$-1 < x \le 1.$$

5. To solve for C, let $x = 0$ and conclude that $C = 0$.
6. Therefore,

$$\ln(x+1) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1},$$

7. $$-1 < x \le 1.$$

Question 10:

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$f(x) = -\frac{1}{(x+1)^2} = \frac{d}{dx}\left[\frac{1}{x+1}\right]$$

1. By taking the first derivative, we have $\displaystyle\frac{d}{dx}\left[\frac{1}{x+1}\right] = \frac{-1}{(x+1)^2}$.
2. Therefore,

$$\frac{-1}{(x+1)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]$$

3. $$= \sum_{n=1}^{\infty}(-1)^n n x^{n-1}$$

4. $$= \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n,$$

5. $$-1 < x < 1.$$

Question 11:

Approximating a real number using Taylor polynomials Use polynomials of order $n = 0, 1, 2,$ and 3 to approximate $\sqrt{18}$.

SOLUTION Letting $f(x) = \sqrt{x}$, we choose the center $a = 16$ because it is near 18, and f and its derivatives are easy to evaluate at 16. The Taylor polynomials have the form

$$p_n(x) = f(16) + f'(16)(x-16) + \frac{f''(16)}{2!}(x-16)^2 + \cdots + \frac{f^{(n)}(16)}{n!}(x-16)^n.$$

We now evaluate the required derivatives:

$$f(x) = \sqrt{x} \Rightarrow f(16) = 4$$

$$f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(16) = \frac{1}{8}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \Rightarrow f''(16) = -\frac{1}{256}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} \Rightarrow f'''(16) = \frac{3}{8192}$$

Therefore, the polynomial p_3 (which includes p_0, p_1, and p_2) is

$$p_3(x) = \underbrace{\underbrace{\underbrace{4}_{p_0} + \frac{1}{8}(x-16)}_{p_1} - \frac{1}{512}(x-16)^2}_{p_2} + \frac{1}{16{,}384}(x-16)^3.$$

Table 9.2

n	Approximations $p_n(18)$	Absolute Error $\lvert\sqrt{18} - p_n(18)\rvert$
0	4	2.43×10^{-1}
1	4.25	7.36×10^{-3}
2	4.242188	4.53×10^{-4}
3	4.242676	3.51×10^{-5}

Question : 12

Find a function represented by the series and give the domain of the function.

$$1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots$$

1. $1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots = \sum_{n=0}^{\infty}\left(\frac{2x}{3}\right)^n$
2. $= \frac{1}{1-(2x/3)}$
3. $= \frac{3}{3-2x}$
4. $-\frac{3}{2} < x < \frac{3}{2}$

Question 13

Find the series representation of the function defined by the integral.

$$\int_0^x \frac{\ln(t+1)}{t}\,dt$$

1. $\frac{1}{1+t} = \sum_{n=0}^{\infty}(-1)^n t^n$
2. $\ln(1+t) = \int \frac{1}{1+t}\,dt$
3. $= \sum_{n=0}^{\infty}\frac{(-1)^n t^{n+1}}{n+1}$
4. $\frac{\ln(t+1)}{t} = \sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n+1}$
5. $\int_0^x \frac{\ln(t+1)}{t}\,dt = \left[\sum_{n=0}^{\infty}\frac{(-1)^n t^{n+1}}{(n+1)^2}\right]_0^x$
6. $= \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{(n+1)^2}$

Question 14

Use the series representation of the function f to find $\lim_{x\to 0} f(x)$ (if it exists).

$$f(x) = \frac{1-\cos x}{x}$$

1. Since $\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$
2. $= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$
3. $1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots$
4. $= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{(2n+2)!}$
5. $\frac{1-\cos}{x} = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \cdots$
6. $= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+2)!}$
7. we have $\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\sum_{n=0}^{\infty}\frac{(-1)x^{2n+1}}{(2n+2)!}$
8. $= 0.$