MATH152 CALCULUS II TUTORIAL – II

(16.10.2015)

Question 1 :

Find a power series for the function, centered at c, and determine the interval of convergence.

$f(x) = \frac{3}{x+2}, \quad c = 0$

1. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{x+2} = \frac{3}{2+x}$$

2. $$= \frac{3/2}{1+(1/2)x}$$

3. $$= \frac{a}{1-r}$$

4. which implies that $a = 3/2$ and $r = (-1/2)x$.
5. Therefore, the power series for $f(x)$ is given by

$$\frac{3}{x+2} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{3}{2}\left(-\frac{1}{2}x\right)^n$$

6. $$= 3\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}}$$

7. $$= \frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n,$$

8. $|x| < 2$
9. or $-2 < x < 2$.

Question 2:

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$f(x) = \ln(x+1) = \int \frac{1}{1+x}\,dx$

1. By integrating, we have $\int \frac{1}{x+1}\,dx = \ln(x+1)$.
2. Therefore,

$$\ln(x+1) = \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx$$

3. $$= C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1},$$

4. $$-1 < x \le 1.$$

5. To solve for C, let $x = 0$ and conclude that $C = 0$.
6. Therefore,

$$\ln(x+1) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1},$$

7. $$-1 < x \le 1.$$

Question 3:

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$f(x) = -\frac{1}{(x+1)^2} = \frac{d}{dx}\left[\frac{1}{x+1}\right]$

1. By taking the first derivative, we have $\frac{d}{dx}\left[\frac{1}{x+1}\right] = \frac{-1}{(x+1)^2}$.
2. Therefore,

$$\frac{-1}{(x+1)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]$$

3. $$= \sum_{n=1}^{\infty}(-1)^n n x^{n-1}$$

4. $$= \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n,$$

5. $$-1 < x < 1.$$

Question 4:

Use the definition to find the Taylor series (centered at c) for the function.

$f(x) = e^{2x}, \ c = 0$

1. For $c = 0$, we have:
2. $f(x) = e^{2x}$
3. $f^{(n)}(x) = 2^n e^{2x}$
4. $\Rightarrow f^{(n)}(0) = 2^n$
5. $e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \cdots$
6. $$= \sum_{n=0}^{\infty}\frac{(2x)^n}{n!}.$$

Question 5:

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$h(x) = \frac{-2}{x^2-1} = \frac{1}{1+x} + \frac{1}{1-x}$$

1. $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$
2. $\frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^n (-x)^n$
3. $= \sum_{n=0}^{\infty} (-1)^{2n} x^n$
4. $= \sum_{n=0}^{\infty} x^n$
5. $h(x) = \frac{-2}{x^2-1}$
6. $= \frac{1}{1+x} + \frac{1}{1-x}$
7. $= \sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n$
8. $= \sum_{n=0}^{\infty} [(-1)^n + 1] x^n$
9. $= 2 + 0x + 2x^2 + 0x^3 + 2x^4 + 0x^5 + 2x^6 + \cdots$
10. $= \sum_{n=0}^{\infty} 2x^{2n},$
11. $-1 < x < 1$
12. (See Exercise 15.)

Question : 6

Find a power series for the function, centered at c, and determine the interval of convergence.

$$g(x) = \frac{3x}{x^2+x-2}, \quad c = 0$$

1. $\frac{3x}{x^2+x-2} = \frac{2}{x+2} + \frac{1}{x-1}$
2. $= \frac{2}{2+x} + \frac{1}{-1+x}$
3. $= \frac{1}{1+(1/2)x} + \frac{-1}{1-x}$
4. Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{3x}{x^2+x-2} = \sum_{n=0}^{\infty}\left(-\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty}(-1)(x)^n$$

5. $= \sum_{n=0}^{\infty}\left[\frac{1}{(-2)^n} - 1\right]x^n.$
6. The interval of convergence is $-1 < x < 1$ since
7. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(1-(-2)^{n+1})x^{n+1}}{(-2)^{n+1}} \cdot \frac{(-2)^n}{(1-(-2)^n)x^n}\right|$
8. $= \lim_{n\to\infty}\left|\frac{(1-(-2)^{n+1})x}{-2-(-2)^{n+1}}\right|$
9. $= |x|.$

Question : 7

Find a function represented by the series and give the domain of the function.

$$1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots$$

1. $1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots = \sum_{n=0}^{\infty}\left(\frac{2x}{3}\right)^n$
2. $= \frac{1}{1-(2x/3)}$
3. $= \frac{3}{3-2x}$
4. $-\frac{3}{2} < x < \frac{3}{2}$

Question : 8

Find a power series for the function, centered at c, and determine the interval of convergence.

$$g(x) = \frac{3x}{x^2 + x - 2}, \quad c = 0$$

1. $\dfrac{3x}{x^2 + x - 2} = \dfrac{2}{x + 2} + \dfrac{1}{x - 1}$
2. $\quad = \dfrac{2}{2 + x} + \dfrac{1}{-1 + x}$
3. $= \dfrac{1}{1 + (1/2)x} + \dfrac{-1}{1 - x}$
4. Writing $f(x)$ as a sum of two geometric series, we have $$\frac{3x}{x^2 + x - 2} = \sum_{n=0}^{\infty}\left(-\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty}(-1)(x)^n$$
5. $\quad = \displaystyle\sum_{n=0}^{\infty}\left[\frac{1}{(-2)^n} - 1\right]x^n.$
6. The interval of convergence is $-1 < x < 1$ since
7. $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(1-(-2)^{n+1})x^{n+1}}{(-2)^{n+1}} \cdot \frac{(-2)^n}{(1-(-2)^n)x^n}\right|$
8. $\quad = \displaystyle\lim_{n\to\infty}\left|\frac{(1-(-2)^{n+1})x}{-2-(-2)^{n+1}}\right|$
9. $\quad = |x|.$

Question 9

Use power series to find the limit (if it exists). Verify the result by using L'Hôpital's Rule.

$$\lim_{x\to0^+}\frac{\arctan x}{\sqrt{x}}$$

1. $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} - \cdots$
2. $\dfrac{\arctan x}{\sqrt{x}} = \sqrt{x} - \dfrac{x^{5/2}}{3} + \dfrac{x^{9/2}}{5} - \dfrac{x^{13/2}}{7} + \dfrac{x^{17/2}}{9} - \cdots$
3. $\displaystyle\lim_{x\to0^+}\frac{\arctan x}{\sqrt{x}} = 0$
4. By L'Hôpital's Rule, $\displaystyle\lim_{x\to0^+}\frac{\arctan x}{\sqrt{x}} = \lim_{x\to0^+}\frac{\left(\frac{1}{1+x^2}\right)}{\left(\frac{1}{2\sqrt{x}}\right)}$
5. $\quad = \displaystyle\lim_{x\to0^+}\frac{2\sqrt{x}}{1+x^2}$
6. $\quad = 0.$

Question 10

Use the series representation of the function f to find $\displaystyle\lim_{x\to0} f(x)$ (if it exists).

$$f(x) = \frac{1-\cos x}{x}$$

1. Since $$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$
2. $\quad = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} - \cdots$
3. $1 - \cos x = \dfrac{x^2}{2!} - \dfrac{x^4}{4!} + \dfrac{x^6}{6!} - \dfrac{x^8}{8!} + \cdots$
4. $\quad = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{(2n+2)!}$
5. $\dfrac{1-\cos}{x} = \dfrac{x}{2!} - \dfrac{x^3}{4!} + \dfrac{x^5}{6!} - \dfrac{x^7}{8!} + \cdots$
6. $\quad = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+2)!}$
7. we have $\displaystyle\lim_{x\to0}\frac{1-\cos x}{x} = \lim_{x\to0}\sum_{n=0}^{\infty}\frac{(-1)x^{2n+1}}{(2n+2)!}$
8. $\quad = 0.$

Question 11

Use power series to approximate the value of the integral with an error of less than 0.0001. (Assume that the integrand is defined as 1 when $x = 0$.)

$$\int_0^1 \frac{\sin x}{x}\,dx$$

1. $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}\right]dx$
2. $\quad = \displaystyle\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}\right]_0^1$
3. $\quad = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(2n+1)!}$
4. Since $1/(7\cdot7!) < 0.0001$, we need three terms:
5. $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx = 1 - \frac{1}{3\cdot3!} + \frac{1}{5\cdot5!} - \cdots$
6. $\quad \approx 0.9461.$
7. (using three nonzero terms)
8. **Note:** We are using $\displaystyle\lim_{x\to0^+}\frac{\sin x}{x} = 1.$

Question 12

Find the series representation of the function defined by the integral.

$$\int_0^x \frac{\ln(t+1)}{t}\,dt$$

1. $$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$$

2. $$\ln(1+t) = \int \frac{1}{1+t}\,dt$$

3. $$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1}$$

4. $$\frac{\ln(t+1)}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n+1}$$

5. $$\int_0^x \frac{\ln(t+1)}{t}\,dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{(n+1)^2}\right]_0^x$$

6. $$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)^2}$$

Question 13

Explain how to use the series

$$g(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

to find the series for each function. Do not find the series.

(a) $f(x) = e^{-x}$
(b) $f(x) = e^{3x}$
(c) $f(x) = xe^x$
(d) $f(x) = e^{2x} + e^{-2x}$

1. (a) Replace x with $(-x)$.
2. (b) Replace x with $3x$.
3. (c) Multiply series by x.
4. (d) Replace x with $2x$,
5. then replace x with $-2x$,
6. and add the two together.