MATH152 CALCULUS II   TUTORIAL – II

(04.03.2016)

Question 1 :

Find the values of x for which the series converges.

$$\sum_{n=0}^{\infty} 2\left(\frac{x}{3}\right)^n$$

1. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2(x/3)^{n+1}}{2(x/3)^n}\right|$
2. $= \lim_{n\to\infty}\left|\frac{x}{3}\right|$
3. $= \left|\frac{x}{3}\right|$
4. For the series to converge: $\left|\frac{x}{3}\right| < 1$
5. $\Longrightarrow -3 < x < 3.$
6. For $x = 3$, the series diverges.
7. For $x = -3$, the series diverges.
8. Answer: $-3 < x < 3$

Question 2 :

Find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

1. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$
2. $= \lim_{n\to\infty}\left|\frac{x}{n+1}\right|$
3. $= 0$
4. The series converges for all x.
5. Therefore, the interval of convergence is $-\infty < x < \infty$.

Question 3 :

Find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$$

1. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-5)^{n+1}}{(n+1)5^{n+1}}\cdot\frac{n5^n}{(-1)^{n+1}(x-5)^n}\right|$
2. $= \lim_{n\to\infty}\left|\frac{n(x-5)}{5(n+1)}\right|$
3. $= \frac{1}{5}|x-5|$
4. $R = 5$
5. Center: $x = 5$
6. Interval: $-5 < x - 5 < 5$ or $0 < x < 10$
7. When $x = 0$, the p-series $\sum_{n=1}^{\infty}\frac{-1}{n}$ diverges.
8. When $x = 10$, the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges.
9. Therefore, the interval of convergence is $0 < x \leq 10$.

Question 4 :

Find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=0}^{\infty} (2n)!\left(\frac{x}{2}\right)^n$$

1. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!x^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(2n)!x^n}\right|$
2. $= \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)x}{2}\right|$
3. $= \infty$
4. Therefore, the series converges only for $x = 0$.

Question 5 :

Find the Maclaurin polynomial of degree n for the function.

$f(x) = e^{-x}, \quad n = 3$

1. $f(x) = e^{-x}$
2. $f(0) = 1$
3. $f'(x) = -e^{-x}$
4. $f'(0) = -1$
5. $f''(x) = e^{-x}$
6. $f''(0) = 1$
7. $f'''(x) = -e^{-x}$
8. $f'''(0) = -1$
9. $P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$
10. $= 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$

Question 6 :

Find the Maclaurin polynomial of degree n for the function.

$f(x) = \sin x, \quad n = 5$

1. $f(x) = \sin x$
2. $f(0) = 0$
3. $f'(x) = \cos x$
4. $f'(0) = 1$
5. $f''(x) = -\sin x$
6. $f''(0) = 0$
7. $f'''(x) = -\cos x$
8. $f'''(0) = -1$
9. $f^{(4)}(x) = \sin x$
10. $f^{(4)}(0) = 0$
11. $f^{(5)}(x) = \cos x$
12. $f^{(5)}(0) = 1$
13. $P_5(x) = 0 + (1)x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$
14. $= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

Question 7 :

Find the nth Taylor polynomial centered at c.

$f(x) = \frac{1}{x}, \quad n = 4, \quad c = 1$

1. $f(x) = \frac{1}{x}$
2. $f(1) = 1$
3. $f'(x) = -\frac{1}{x^2}$
4. $f'(1) = -1$
5. $f''(x) = \frac{2}{x^3}$
6. $f''(1) = 2$
7. $f'''(x) = -\frac{6}{x^4}$
8. $f'''(1) = -6$
9. $f^{(4)}(x) = \frac{24}{x^5}$
10. $f^{(4)}(1) = 24$
11. $P_4(x) = 1 - (x-1) + \frac{2}{2!}(x-1)^2 + \frac{-6}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$
12. $= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4$

Question 8:

Find a power series for the function, centered at c, and determine the interval of convergence.

$$f(x) = \frac{3}{x+2}, \quad c = 0$$

1. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{x+2} = \frac{3}{2+x}$$

2. $$= \frac{3/2}{1+(1/2)x}$$

3. $$= \frac{a}{1-r}$$

4. which implies that $a = 3/2$ and $r = (-1/2)x$.
5. Therefore, the power series for $f(x)$ is given by

$$\frac{3}{x+2} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{3}{2}\left(-\frac{1}{2}x\right)^n$$

6. $$= 3\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}}$$

7. $$= \frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n,$$

8. $|x| < 2$
9. or $-2 < x < 2$.

Question 9:

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$f(x) = \ln(x+1) = \int \frac{1}{1+x}\,dx$$

1. By integrating, we have $\int \frac{1}{x+1}\,dx = \ln(x+1)$.
2. Therefore,

$$\ln(x+1) = \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx$$

3. $$= C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1},$$

4. $$-1 < x \le 1.$$

5. To solve for C, let $x = 0$ and conclude that $C = 0$.
6. Therefore,

$$\ln(x+1) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1},$$

7. $$-1 < x \le 1.$$

Question 10:

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$f(x) = -\frac{1}{(x+1)^2} = \frac{d}{dx}\left[\frac{1}{x+1}\right]$$

1. By taking the first derivative,

   we have $\frac{d}{dx}\left[\frac{1}{x+1}\right] = \frac{-1}{(x+1)^2}$.

2. Therefore,

$$\frac{-1}{(x+1)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]$$

3. $$= \sum_{n=1}^{\infty}(-1)^n n x^{n-1}$$

4. $$= \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n,$$

5. $$-1 < x < 1.$$

Question 11:

Approximating a real number using Taylor polynomials Use polynomials of order $n = 0, 1, 2,$ and 3 to approximate $\sqrt{18}$.

SOLUTION Letting $f(x) = \sqrt{x}$, we choose the center $a = 16$ because it is near 18, and f and its derivatives are easy to evaluate at 16. The Taylor polynomials have the form

$$p_n(x) = f(16) + f'(16)(x-16) + \frac{f''(16)}{2!}(x-16)^2 + \cdots + \frac{f^{(n)}(16)}{n!}(x-16)^n.$$

We now evaluate the required derivatives:

$$f(x) = \sqrt{x} \Rightarrow f(16) = 4$$

$$f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(16) = \frac{1}{8}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \Rightarrow f''(16) = -\frac{1}{256}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} \Rightarrow f'''(16) = \frac{3}{8192}$$

Therefore, the polynomial p_3 (which includes p_0, p_1, and p_2) is

$$p_3(x) = \underbrace{\underbrace{\underbrace{4}_{p_0} + \frac{1}{8}(x-16)}_{p_1} - \frac{1}{512}(x-16)^2}_{p_2} + \frac{1}{16{,}384}(x-16)^3.$$

Table 9.2

n	Approximations $p_n(18)$	Absolute Error $\lvert\sqrt{18} - p_n(18)\rvert$
0	4	2.43×10^{-1}
1	4.25	7.36×10^{-3}
2	4.242188	4.53×10^{-4}
3	4.242676	3.51×10^{-5}

Question : 12

Find a function represented by the series and give the domain of the function.

$$1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots$$

1. $1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots = \sum_{n=0}^{\infty}\left(\frac{2x}{3}\right)^n$
2. $= \frac{1}{1-(2x/3)}$
3. $= \frac{3}{3-2x}$
4. $-\frac{3}{2} < x < \frac{3}{2}$

Question 13

Find the series representation of the function defined by the integral.

$$\int_0^x \frac{\ln(t+1)}{t}\,dt$$

1. $\frac{1}{1+t} = \sum_{n=0}^{\infty}(-1)^n t^n$
2. $\ln(1+t) = \int \frac{1}{1+t}\,dt$
3. $= \sum_{n=0}^{\infty}\frac{(-1)^n t^{n+1}}{n+1}$
4. $\frac{\ln(t+1)}{t} = \sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n+1}$
5. $\int_0^x \frac{\ln(t+1)}{t}\,dt = \left[\sum_{n=0}^{\infty}\frac{(-1)^n t^{n+1}}{(n+1)^2}\right]_0^x$
6. $= \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{(n+1)^2}$

Question 14

Use the series representation of the function f to find $\lim_{x\to 0} f(x)$ (if it exists).

$$f(x) = \frac{1-\cos x}{x}$$

1. Since $\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$
2. $= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$
3. $1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots$
4. $= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{(2n+2)!}$
5. $\frac{1-\cos}{x} = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \cdots$
6. $= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+2)!}$
7. we have $\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\sum_{n=0}^{\infty}\frac{(-1)x^{2n+1}}{(2n+2)!}$
8. $= 0.$