

MATH152 CALCULUS II TUTORIAL – II

(04.10.2019)

Question 1 :

Find the values of x for which the series converges.

$$\sum_{n=0}^{\infty} 2\left(\frac{x}{3}\right)^n$$

1. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2(x/3)^{n+1}}{2(x/3)^n} \right|$

2. $= \lim_{n \rightarrow \infty} \left| \frac{x}{3} \right|$

3. $= \left| \frac{x}{3} \right|$

4. For the series to converge: $\left| \frac{x}{3} \right| < 1$

5. $\implies -3 < x < 3$.

6. For $x = 3$, the series diverges.

7. For $x = -3$, the series diverges.

8. Answer: $-3 < x < 3$

Question 2 :

Find the values of x for which the series converges.

$$\sum_{n=0}^{\infty} n! \left(\frac{x}{2}\right)^n$$

1. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)! |x/2|^{n+1}}{n! |x/2|^n}$

2. $= \lim_{n \rightarrow \infty} (n+1) \left| \frac{x}{2} \right|$

3. $= \infty$

4. The series converges only at $x = 0$.

Question 3 :

Find the values of x for which the series converges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{n}$$

1. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x+1)^{n+1}/(n+1)}{x^n/n} \right|$

2. $= \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} (x+1) \right|$

3. $= |x+1|$

4. For the series to converge,

$$|x+1| < 1$$

5. $\implies -1 < x+1 < 1$

6. $\implies -2 < x < 0$.

7. For $x = 0$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$,

8. converges.

9. For $x = -2$, $\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n}$

10. $= \sum_{n=1}^{\infty} \frac{1}{n}$,

11. diverges.

12. Answer: $-2 < x \leq 0$

Question 4 :

Find the interval of convergence of the power series.
(Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=0}^{\infty} n! (x-2)^n$$

1. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)! (x-2)^{n+1}}{n! (x-2)^n} \right|$

2. $= \infty$

3. which implies that the series converges only at the center $x = 2$.

Question 5 :

a) Find the Maclaurin series of $f(x) = \ln(1+x^2)$ in powers of x .

b) Find the Taylor series of $f(x) = \frac{1}{x}$ in powers of $x-1$.

$$a) \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \text{ (converges for } |x| < 1) \Rightarrow \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \Rightarrow \frac{2x}{1+x^2} = 2 \sum_{n=0}^{\infty} (-1)^n x^{2n+1}$$

$$\Rightarrow \int \frac{2x}{1+x^2} dx = 2 \sum_{n=0}^{\infty} (-1)^n \int x^{2n+1} dx \Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n+1} \Rightarrow x^2 \sum_{n=0}^{\infty} (-1)^n x^{2n} \Rightarrow$$

$$\ln(1+x^2) + c = x^2 \sum_{n=0}^{\infty} (-1)^n x^{2n} \Rightarrow \text{take } x=0 \Rightarrow \ln 1 + c = 0 \Rightarrow c = 0$$

$$\Rightarrow \ln(1+x^2) = x^2 \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad -1 < x < 1,$$

$$b) \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \Rightarrow \frac{1}{x} = \frac{1}{1+(x-1)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \Rightarrow \text{(converges for } |x-1| < 1)$$

Question 6:

a) Find the Maclaurin series of $f(x) = \ln(1+x^2)$ in powers of x .

b) Find the Taylor series of $f(x) = \frac{1}{x}$ in powers of $x-1$.

$$a) \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \text{ (converges for } |x| < 1) \Rightarrow \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \Rightarrow \frac{2x}{1+x^2} = 2 \sum_{n=0}^{\infty} (-1)^n x^{2n+1}$$

$$\Rightarrow \int \frac{2x}{1+x^2} dx = 2 \sum_{n=0}^{\infty} (-1)^n \int x^{2n+1} dx \Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n+1} \Rightarrow x^2 \sum_{n=0}^{\infty} (-1)^n x^{2n} \Rightarrow$$

$$\ln(1+x^2) + c = x^2 \sum_{n=0}^{\infty} (-1)^n x^{2n} \Rightarrow \text{take } x=0 \Rightarrow \ln 1 + c = 0 \Rightarrow c = 0$$

$$\Rightarrow \ln(1+x^2) = x^2 \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad -1 < x < 1,$$

$$b) \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \Rightarrow \frac{1}{x} = \frac{1}{1+(x-1)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \Rightarrow \text{(converges for } |x-1| < 1)$$

Question 7:

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$f(x) = -\frac{1}{(x+1)^2} = \frac{d}{dx} \left[\frac{1}{x+1} \right]$$

1. By taking the first derivative,

$$\text{we have } \frac{d}{dx} \left[\frac{1}{x+1} \right] = \frac{-1}{(x+1)^2}.$$

2. Therefore,

$$\frac{-1}{(x+1)^2} = \frac{d}{dx} \left[\sum_{n=0}^{\infty} (-1)^n x^n \right]$$

$$3. \quad = \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$$

$$4. \quad = \sum_{n=0}^{\infty} (-1)^{n+1} (n+1) x^n,$$

$$5. \quad -1 < x < 1.$$

Question 8:

Find a power series for the function, centered at c , and determine the interval of convergence.

$$f(x) = \frac{2}{1-x^2}, \quad c = 0$$

$$1. \quad \frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$$

2. Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{2}{1-x^2} = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (-x)^n$$

$$3. \quad = \sum_{n=0}^{\infty} (1 + (-1)^n) x^n$$

$$4. \quad = \sum_{n=0}^{\infty} 2x^{2n}.$$

5. The interval of convergence is $|x^2| < 1$

6. or $-1 < x < 1$ since

$$7. \quad \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2x^{2n+2}}{2x^{2n}} \right|$$

$$8. \quad = |x^2|.$$

Question 9:

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$f(x) = \ln(x+1) = \int \frac{1}{x+1} dx$$

1. By integrating, we have $\int \frac{1}{x+1} dx = \ln(x+1)$.

2. Therefore,

$$\ln(x+1) = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] dx$$

$$3. \quad = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1},$$

$$4. \quad -1 < x \leq 1.$$

5. To solve for C , let $x = 0$ and conclude that $C = 0$.

6. Therefore,

$$\ln(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1},$$

$$7. \quad -1 < x \leq 1.$$

Question 10:

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$f(x) = -\frac{1}{(x+1)^2} = \frac{d}{dx} \left[\frac{1}{x+1} \right]$$

1. By taking the first derivative,

$$\text{we have } \frac{d}{dx} \left[\frac{1}{x+1} \right] = \frac{-1}{(x+1)^2}.$$

2. Therefore,

$$\frac{-1}{(x+1)^2} = \frac{d}{dx} \left[\sum_{n=0}^{\infty} (-1)^n x^n \right]$$

$$3. \quad = \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$$

$$4. \quad = \sum_{n=0}^{\infty} (-1)^{n+1} (n+1) x^n,$$

$$5. \quad -1 < x < 1.$$

Question 11:

Approximating a real number using Taylor polynomials Use polynomials of order $n = 0, 1, 2,$ and 3 to approximate $\sqrt{18}$.

SOLUTION Letting $f(x) = \sqrt{x}$, we choose the center $a = 16$ because it is near 18, and f and its derivatives are easy to evaluate at 16. The Taylor polynomials have the form

$$p_n(x) = f(16) + f'(16)(x-16) + \frac{f''(16)}{2!}(x-16)^2 + \cdots + \frac{f^{(n)}(16)}{n!}(x-16)^n.$$

We now evaluate the required derivatives:

$$f(x) = \sqrt{x} \Rightarrow f(16) = 4$$

$$f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(16) = \frac{1}{8}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \Rightarrow f''(16) = -\frac{1}{256}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} \Rightarrow f'''(16) = \frac{3}{8192}$$

Therefore, the polynomial p_3 (which includes $p_0, p_1,$ and p_2) is

$$p_3(x) = \underbrace{4}_{p_0} + \underbrace{\frac{1}{8}(x-16)}_{p_1} - \underbrace{\frac{1}{512}(x-16)^2 + \frac{1}{16,384}(x-16)^3}_{p_2}$$

Table 9.2

n	Approximations $p_n(18)$	Absolute Error $ \sqrt{18} - p_n(18) $
0	4	2.43×10^{-1}
1	4.25	7.36×10^{-3}
2	4.242188	4.53×10^{-4}
3	4.242676	3.51×10^{-5}

Question 12

Find a function represented by the series and give the domain of the function.

$$1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \dots$$

1. $1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \dots = \sum_{n=0}^{\infty} \left(\frac{2x}{3}\right)^n$
2. $= \frac{1}{1 - (2x/3)}$
3. $= \frac{3}{3 - 2x}$
4. $-\frac{3}{2} < x < \frac{3}{2}$

Question 13

Find the series representation of the function defined by the integral.

$$\int_0^x \frac{\ln(t+1)}{t} dt$$

1. $\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$
2. $\ln(1+t) = \int \frac{1}{1+t} dt$
3. $= \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1}$
4. $\frac{\ln(t+1)}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n+1}$
5. $\int_0^x \frac{\ln(t+1)}{t} dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{(n+1)^2} \right]_0^x$
6. $= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)^2}$

Question 14

Use the series representation of the function f to find $\lim_{x \rightarrow 0} f(x)$ (if it exists).

$$f(x) = \frac{1 - \cos x}{x}$$

1. Since

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$2. = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$3. 1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \dots$$

$$\frac{x^7}{8!} + \dots$$

$$7. \text{ we have } \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = \lim_{x \rightarrow 0} \sum_{n=0}^{\infty} \frac{(-1)^{2n+1} x^{2n+1}}{(2n+2)!}$$

$$8. = 0.$$