

MATH152 CALCULUS II TUTORIAL – II

(07.10.2016)

Question 1 :

Find the values of x for which the series converges.

$$\sum_{n=0}^{\infty} 2 \left(\frac{x}{3}\right)^n$$

$$1. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2(x/3)^{n+1}}{2(x/3)^n} \right|$$

$$2. = \lim_{n \rightarrow \infty} \left| \frac{x}{3} \right|$$

$$3. = \left| \frac{x}{3} \right|$$

$$4. \text{ For the series to converge: } \left| \frac{x}{3} \right| < 1$$

$$5. \implies -3 < x < 3.$$

6. For $x = 3$, the series diverges.

7. For $x = -3$, the series diverges.

8. Answer: $-3 < x < 3$

Question 2 :

Find the interval of convergence of the power series.
(Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$1. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$$2. = \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right|$$

$$3. = 0$$

4. The series converges for all x .

5. Therefore, the interval of convergence is $-\infty < x < \infty$.

Question 3 :

Find the interval of convergence of the power series.
(Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$$

$$1. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(x-5)^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n5^n}{(-1)^{n+1}(x-5)^n} \right|$$

$$2. = \lim_{n \rightarrow \infty} \left| \frac{n(x-5)}{5(n+1)} \right|$$

$$3. = \frac{1}{5}|x-5|$$

4. $R = 5$

5. Center: $x = 5$

6. Interval: $-5 < x - 5 < 5$ or $0 < x < 10$

7. When $x = 0$, the p -series $\sum_{n=1}^{\infty} \frac{-1}{n}$ diverges.

8. When $x = 10$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

9. Therefore, the interval of convergence is $0 < x \leq 10$.

Question 4 :

Find the interval of convergence of the power series.
(Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=0}^{\infty} (2n)! \left(\frac{x}{2}\right)^n$$

$$1. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)!x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(2n)!x^n} \right|$$

$$2. = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)x}{2} \right|$$

$$3. = \infty$$

4. Therefore, the series converges only for $x = 0$.

Question 5 :

Find the Maclaurin polynomial of degree n for the function.

$$f(x) = e^{-x}, \quad n = 3$$

1. $f(x) = e^{-x}$
2. $f(0) = 1$
3. $f'(x) = -e^{-x}$
4. $f'(0) = -1$
5. $f''(x) = e^{-x}$
6. $f''(0) = 1$
7. $f'''(x) = -e^{-x}$
8. $f'''(0) = -1$
9. $P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$
10. $= 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$

Question 6 :

Find the Maclaurin polynomial of degree n for the function.

$$f(x) = \sin x, \quad n = 5$$

1. $f(x) = \sin x$
2. $f(0) = 0$
3. $f'(x) = \cos x$
4. $f'(0) = 1$
5. $f''(x) = -\sin x$
6. $f''(0) = 0$
7. $f'''(x) = -\cos x$
8. $f'''(0) = -1$
9. $f^{(4)}(x) = \sin x$
10. $f^{(4)}(0) = 0$
11. $f^{(5)}(x) = \cos x$
12. $f^{(5)}(0) = 1$
13. $P_5(x) = 0 + (1)x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$
14. $= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

Question 7 :

Find the n th Taylor polynomial centered at c .

$$f(x) = \frac{1}{x}, \quad n = 4, \quad c = 1$$

1. $f(x) = \frac{1}{x}$
2. $f(1) = 1$
3. $f'(x) = -\frac{1}{x^2}$
4. $f'(1) = -1$
5. $f''(x) = \frac{2}{x^3}$
6. $f''(1) = 2$
7. $f'''(x) = -\frac{6}{x^4}$
8. $f'''(1) = -6$
9. $f^{(4)}(x) = \frac{24}{x^5}$
10. $f^{(4)}(1) = 24$
11. $P_4(x) = 1 - (x - 1) + \frac{2}{2!}(x - 1)^2 + \frac{-6}{3!}(x - 1)^3 + \frac{24}{4!}(x - 1)^4$
12. $= 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + (x - 1)^4$

Question 8:

Find a power series for the function, centered at c , and determine the interval of convergence.

$$f(x) = \frac{3}{x+2}, \quad c = 0$$

1. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{x+2} = \frac{3}{2+x}$$

$$2. \quad = \frac{3/2}{1+(1/2)x}$$

$$3. \quad = \frac{a}{1-r}$$

4. which implies that $a = 3/2$ and $r = (-1/2)x$.

5. Therefore, the power series for $f(x)$ is given by

$$\frac{3}{x+2} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{3}{2} \left(-\frac{1}{2}x\right)^n$$

$$6. \quad = 3 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}}$$

$$7. \quad = \frac{3}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n,$$

$$8. \quad |x| < 2$$

$$9. \quad \text{or } -2 < x < 2.$$

Question 9:

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$f(x) = \ln(x+1) = \int \frac{1}{1+x} dx$$

$$1. \quad \text{By integrating, we have } \int \frac{1}{x+1} dx = \ln(x+1).$$

2. Therefore,

$$\ln(x+1) = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] dx$$

$$3. \quad = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1},$$

$$4. \quad -1 < x \leq 1.$$

5. To solve for C , let $x = 0$ and conclude that $C = 0$.

6. Therefore,

$$\ln(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1},$$

$$7. \quad -1 < x \leq 1.$$

Question 10:

Use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$f(x) = -\frac{1}{(x+1)^2} = \frac{d}{dx} \left[\frac{1}{x+1} \right]$$

1. By taking the first derivative,

$$\text{we have } \frac{d}{dx} \left[\frac{1}{x+1} \right] = \frac{-1}{(x+1)^2}.$$

2. Therefore,

$$\frac{-1}{(x+1)^2} = \frac{d}{dx} \left[\sum_{n=0}^{\infty} (-1)^n x^n \right]$$

$$3. \quad = \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$$

$$4. \quad = \sum_{n=0}^{\infty} (-1)^{n+1} (n+1) x^n,$$

$$5. \quad -1 < x < 1.$$

Question 11:

Approximating a real number using Taylor polynomials Use polynomials of order $n = 0, 1, 2$, and 3 to approximate $\sqrt{18}$.

SOLUTION Letting $f(x) = \sqrt{x}$, we choose the center $a = 16$ because it is near 18, and its derivatives are easy to evaluate at 16. The Taylor polynomials have the form

$$p_n(x) = f(16) + f'(16)(x-16) + \frac{f''(16)}{2!}(x-16)^2 + \cdots + \frac{f^{(n)}(16)}{n!}(x-16)^n.$$

We now evaluate the required derivatives:

$$f(x) = \sqrt{x} \Rightarrow f(16) = 4$$

$$f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(16) = \frac{1}{8}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \Rightarrow f''(16) = -\frac{1}{256}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} \Rightarrow f'''(16) = \frac{3}{8192}$$

Therefore, the polynomial p_3 (which includes p_0, p_1 , and p_2) is

$$p_3(x) = \underbrace{\frac{4}{1} + \frac{1}{8}(x-16)}_{p_1} - \frac{1}{512}(x-16)^2 + \frac{1}{16,384}(x-16)^3.$$

$\underbrace{\hspace{10em}}_{p_2}$

Table 9.2

n	Approximations $p_n(18)$	Absolute Error $ \sqrt{18} - p_n(18) $
0	4	2.43×10^{-1}
1	4.25	7.36×10^{-3}
2	4.242188	4.53×10^{-4}
3	4.242676	3.51×10^{-5}

Question : 12

Find a function represented by the series and give the domain of the function.

$$1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \dots$$

- $1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \dots = \sum_{n=0}^{\infty} \left(\frac{2x}{3}\right)^n$
- $= \frac{1}{1 - (2x/3)}$
- $= \frac{3}{3 - 2x}$
- $-\frac{3}{2} < x < \frac{3}{2}$

Question 13

Find the series representation of the function defined by the integral.

$$\int_0^x \frac{\ln(t+1)}{t} dt$$

- $\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$
- $\ln(1+t) = \int \frac{1}{1+t} dt$
- $= \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1}$
- $\frac{\ln(t+1)}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n+1}$
- $\int_0^x \frac{\ln(t+1)}{t} dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{(n+1)^2} \right]_0^x$
- $= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)^2}$

Question 14

Use the series representation of the function f to find $\lim_{x \rightarrow 0} f(x)$ (if it exists).

$$f(x) = \frac{1 - \cos x}{x}$$

1. Since

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$2. = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$3. 1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \dots$$

$$4. = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!}$$

$$5. \frac{1 - \cos x}{x} = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \dots$$

$$6. = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+2)!}$$

$$7. \text{ we have } \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = \lim_{x \rightarrow 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+2)!}$$

$$8. = 0.$$