

# MATH152 CALCULUS II TUTORIAL – 12

(18.05.2018)

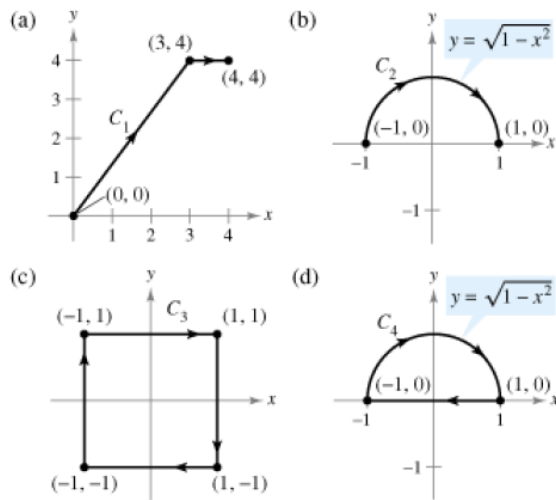
**Question 1:** Let  $F(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$

Find the value of the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(Hint: If  $\mathbf{F}$  is conservative, the integration may be easier on an alternative path.)

$$\int_C y^2 dx + 2xy dy$$



1. Since  $\partial M/\partial y = \partial N/\partial x = 2y$ ,  $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$  is conservative.
2. The potential function is  $f(x, y) = xy^2 + k$ .
3. Therefore, we can use the Fundamental Theorem of Line Integrals.
4. (a)  $\int_C y^2 dx + 2xy dy = \left[ xy^2 \right]_{(0,0)}^{(4,4)}$
5.  $= 64$
6. (b)  $\int_C y^2 dx + 2xy dy = \left[ xy^2 \right]_{(-1,0)}^{(1,0)}$
7.  $= 0$
8. (c) and (d) Since  $C$  is a closed curve,

$$\int_C y^2 dx + 2xy dy = 0.$$

**Question 2 :**

Find the value of the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(Hint: If  $\mathbf{F}$  is conservative, the integration may be easier on an alternative path.)

$$\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

(a)  $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 4$

(b)  $\mathbf{r}_2(t) = t^2\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}, 0 \leq t \leq 2$

1. Since  $\text{curl } \mathbf{F} = \mathbf{0}$ ,  $\mathbf{F}(x, y, z)$  is conservative.
2. The potential function is  $f(x, y, z) = xyz + k$ .

3. (a)  $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 4$

4.  $\int_C \mathbf{F} \cdot d\mathbf{r} = \left[ xyz \right]_{(0, 2, 0)}^{(4, 2, 4)}$

5.  $= 32$

**Question 3 :**

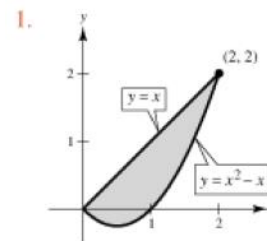
Use Green's Theorem to evaluate the integral

$$\int_C (y - x) dx + (2x - y) dy$$

for the given path.

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1.$$

C: boundary of the region lying between the graphs of  $y = x$  and  $y = x^2 - x$



$$2. \int_C (y - x) dx + (2x - y) dy = \int_0^2 \int_{x^2-x}^x dy dx$$

$$3. = \int_0^2 (2x - x^2) dx$$

$$4. = \frac{4}{3}$$

#### Question 4:

Use Green's Theorem to evaluate the line integral.

$$\int_C \sin x \cos y dx + (xy + \cos x \sin y) dy$$

C: boundary of the region lying between the graphs of  $y = x$  and  $y = \sqrt{x}$

1. By Green's Theorem,

$$\int_C \sin x \cos y dx + (xy + \cos x \sin y) dy = \iint_R [(y - \sin x \sin y) - (-\sin x$$

$$2. = \int_0^1 \int_x^{\sqrt{x}} y dy dx$$

$$3. = \frac{1}{2} \int_0^1 (x - x^2) dx$$

$$4. = \frac{1}{2} \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$5. = \frac{1}{12}$$

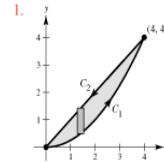
#### Question 5:

Verify Green's Theorem by evaluating both integrals

$$\int_C y^2 dx + x^2 dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

for the given path.

C: boundary of the region lying between the graphs of  $y = x$  and  $y = x^2/4$



$$1. \mathbf{r}(t) = \begin{cases} ti + t^2/4\mathbf{j}, & 0 \leq t \leq 4 \\ (8-t)\mathbf{i} + (8-t)\mathbf{j}, & 4 \leq t \leq 8 \end{cases}$$

$$3. \int_C y^2 dx + x^2 dy = \int_0^4 \left[ \frac{t^4}{16}(dt) + t^2 \left( \frac{t}{2} dt \right) \right] + \int_4^8 [(8-t)^2(-dt) + (8-t)^2(-dt)]$$

$$4. = \int_0^4 \left[ \frac{t^4}{16} + \frac{t^3}{2} \right] dt + \int_4^8 -2(8-t)^2 dt$$

$$5. = \frac{224}{5} - \frac{128}{3}$$

$$6. = \frac{32}{15}$$

$$7. \text{ By Green's Theorem,}$$

$$\iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \int_0^4 \int_{x^2/4}^x (2x - 2y) dy dx$$

$$8. = \int_0^4 \left( x^2 - \frac{x^3}{2} + \frac{x^4}{16} \right) dx$$

$$9. = \frac{32}{15}$$

**Question : 6**

Use Green's Theorem to evaluate the integral

$$\int_C (y - x) dx + (2x - y) dy$$

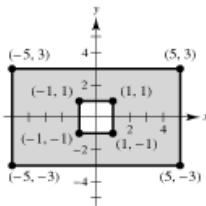
for the given path.

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1.$$

C: boundary of the region lying inside the rectangle bounded by  $x = -5, x = 5, y = -3, \text{ and } y = 3,$  and outside the square bounded by  $x = -1, x = 1, y = -1, \text{ and } y = 1$

1. From the accompanying figure, we see that  $R$  is the shaded region.

2.



3. Thus, Green's Theorem yields

$$\int_C (y - x) dx + (2x - y) dy = \iint_R 1 dA$$

$$= \text{Area of } R$$

$$= 6(10) - 2(2)$$

$$= 56.$$

**Question : 7**

Evaluate  $\iint_S xy dS$ .

S:  $z = 6 - x - 2y$ , first octant

$$1. \frac{\partial z}{\partial x} = -1$$

$$2. \frac{\partial z}{\partial y} = -2$$

$$3. \iint_S xy dS = \int_0^6 \int_0^{3-(x/2)} xy \sqrt{1 + (-1)^2 + (-2)^2} dy dx$$

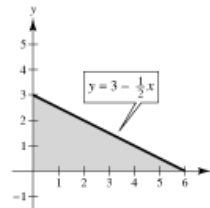
$$4. = \sqrt{6} \int_0^6 \left[ \frac{xy^2}{2} \right]_0^{3-(x/2)} dx$$

$$5. = \frac{\sqrt{6}}{2} \int_0^6 x \left( 9 - 3x + \frac{1}{4}x^2 \right) dx$$

$$6. = \frac{\sqrt{6}}{2} \left[ \frac{9x^2}{2} - x^3 + \frac{x^4}{16} \right]_0^6$$

$$7. = \frac{27\sqrt{6}}{2}$$

8.



### Question : 8

Evaluate  $\iint_S (x - 2y + z) dS$ .

$S: z = 4 - x, 0 \leq x \leq 4, 0 \leq y \leq 4$

1.  $\frac{\partial z}{\partial x} = -1$

2.  $\frac{\partial z}{\partial y} = 0$

3.  $\iint_S (x - 2y + z) dS = \int_0^4 \int_0^4 (x - 2y + 4 - x) \sqrt{1 + (-1)^2 + (0)^2} dy dx$

4.  $= \sqrt{2} \int_0^4 \int_0^4 (4 - 2y) dy dx$

5.  $= 0$

### Question : 9

Find the flux of  $\mathbf{F}$  through  $S$ ,

$$\iint_S \mathbf{F} \cdot \mathbf{N} dS$$

where  $\mathbf{N}$  is the upward unit normal vector to  $S$ .

$$\mathbf{F}(x, y, z) = 3z\mathbf{i} - 4y\mathbf{j} + y\mathbf{k}$$

$S: x + y + z = 1$ , first octant

1.  $G(x, y, z) = x + y + z - 1$

2.  $\nabla G(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

3.  $\iint_S \mathbf{F} \cdot \mathbf{N} dS = \iint_R \mathbf{F} \cdot \nabla G dA$

4.  $= \int_0^1 \int_0^{1-x} (3z - 4 + y) dy dx$

5.  $= \int_0^1 \int_0^{1-x} [3(1 - x - y) - 4 + y] dy dx$

6.  $= \int_0^1 \int_0^{1-x} (-1 - 3x - 2y) dy dx$

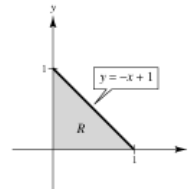
7.  $= \int_0^1 \left[ -y - 3xy - y^2 \right]_0^{1-x} dx$

8.  $= - \int_0^1 [(1 - x) + 3x(1 - x) + (1 - x)^2] dx$

9.  $= - \int_0^1 (2 - 2x^2) dx$

10.  $= -\frac{4}{3}$

11.



### Question : 10

Use the Divergence Theorem to evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, dS$$

and find the outward flux of  $\mathbf{F}$  through the surface of the solid bounded by the graphs of the equations. Use a computer algebra system to verify your results.

$$\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$$

$$S: x = 0, x = a, y = 0, y = a, z = 0, z = a$$

1. Since  $\text{div } \mathbf{F} = 2x + 2y + 2z$ , we have

$$2. \iiint_Q \text{div } \mathbf{F} \, dV = \int_0^a \int_0^a \int_0^a (2x + 2y + 2z) \, dz \, dy \, dx$$

$$3. = \int_0^a \int_0^a (2ax + 2ay + a^2) \, dy \, dx$$

$$4. = \int_0^a (2a^2x + 2a^3) \, dx$$

$$5. = \left[ a^2x^2 + 2a^3x \right]_0^a$$

$$6. = 3a^4.$$

### Question : 11

Verify Stokes's Theorem by evaluating

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot d\mathbf{r}$$

as a line integral and as a double integral.

$$\mathbf{F}(x, y, z) = (-y + z)\mathbf{i} + (x - z)\mathbf{j} + (x - y)\mathbf{k}$$

$$S: z = \sqrt{1 - x^2 - y^2}$$

1. In this case,  $M = -y + z$ ,

2.  $N = x - z$ ,

3.  $P = x - y$  and

4.  $C$  is the circle  $x^2 + y^2 = 1$ ,

5.  $z = 0$ ,

6.  $dz = 0$ .

$$7. \text{Line Integral: } \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (-y + z) \, dx + (x - z) \, dy + (x - y) \, dz$$

$$8. = \int_C -y \, dx + x \, dy$$

9. Letting  $x = \cos t$ ,

10.  $y = \sin t$ , we have

11.  $dx = -\sin t \, dt$ ,

12.  $dy = \cos t \, dt$  and

$$13. \int_C -y \, dx + x \, dy = \int_0^{2\pi} (\sin^2 t + \cos^2 t) \, dt$$

14.  $= 2\pi$ .

15. **Double Integral:** Consider  $F(x, y, z) = x^2 + y^2 + z^2 - 1$ . Then

$$16. \mathbf{N} = \frac{\nabla F}{\|\nabla F\|}$$

$$17. = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}}$$

$$18. = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

19. Since  $z^2 = 1 - x^2 - y^2$ ,

$$20. z_x = \frac{-2x}{2z}$$

$$21. = \frac{-x}{z}, \text{ and}$$

$$22. z_y = \frac{-y}{z},$$

$$23. dS = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} \, dA$$

$$24. = \frac{1}{z} \, dA.$$

25. Now, since  $\text{curl } \mathbf{F} = 2\mathbf{k}$ , we have

$$26. \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS = \iint_R 2z \left( \frac{1}{z} \right) \, dA$$

$$27. = \iint_R 2 \, dA$$

$$28. = 2(\text{Area of circle of radius 1})$$

$$29. = 2\pi.$$

