

MATH152 CALCULUS II TUTORIAL – 12

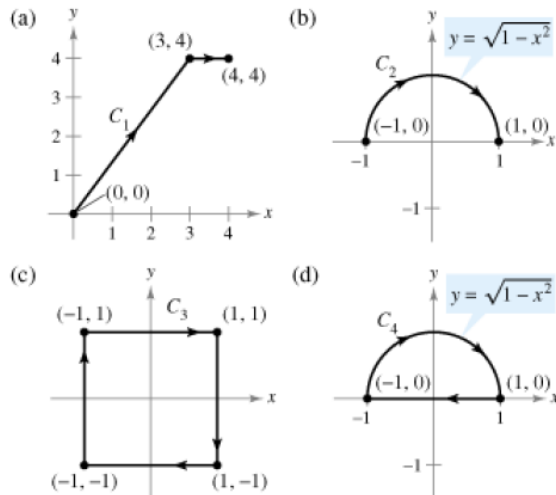
Question 1: Let $F(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$

Find the value of the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(Hint: If \mathbf{F} is conservative, the integration may be easier on an alternative path.)

$$\int_C y^2 dx + 2xy dy$$



1. Since $\partial M/\partial y = \partial N/\partial x = 2y$, $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$ is conservative.
 2. The potential function is $f(x, y) = xy^2 + k$.
 3. Therefore, we can use the Fundamental Theorem of Line Integrals.
 4. (a) $\int_C y^2 dx + 2xy dy = \left[x^2 y \right]_{(0,0)}^{(4,4)}$
 5. $= 64$
 6. (b) $\int_C y^2 dx + 2xy dy = \left[x^2 y \right]_{(-1,0)}^{(1,0)}$
 7. $= 0$
 8. (c) and (d) Since C is a closed curve,
- $$\int_C y^2 dx + 2xy dy = 0.$$

Question 2 :

Find the value of the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(Hint: If \mathbf{F} is conservative, the integration may be easier on an alternative path.)

$$\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 4$

(b) $\mathbf{r}_2(t) = t^2\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}, 0 \leq t \leq 2$

1. Since $\text{curl } \mathbf{F} = \mathbf{0}$, $\mathbf{F}(x, y, z)$ is conservative.
2. The potential function is $f(x, y, z) = xyz + k$.
3. (a) $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 4$
4. $\int_C \mathbf{F} \cdot d\mathbf{r} = \left[xyz \right]_{(0,2,0)}^{(4,2,4)}$
5. $= 32$

Question 3 :

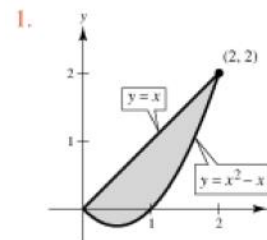
Use Green's Theorem to evaluate the integral

$$\int_C (y - x) dx + (2x - y) dy$$

for the given path.

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1.$$

C: boundary of the region lying between the graphs of $y = x$ and $y = x^2 - x$



1.

$$2. \int_C (y - x) dx + (2x - y) dy = \int_0^2 \int_{x^2-x}^x dy dx$$

$$3. = \int_0^2 (2x - x^2) dx$$

$$4. = \frac{4}{3}$$

Question 4:

Use Green's Theorem to evaluate the line integral.

$$\int_C \sin x \cos y dx + (xy + \cos x \sin y) dy$$

C: boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$

1. By Green's Theorem,

$$\int_C \sin x \cos y dx + (xy + \cos x \sin y) dy = \iint_R [(y - \sin x \sin y) - (-\sin x$$

$$2. = \int_0^1 \int_x^{\sqrt{x}} y dy dx$$

$$3. = \frac{1}{2} \int_0^1 (x - x^2) dx$$

$$4. = \frac{1}{2} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$5. = \frac{1}{12}$$

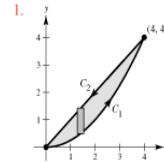
Question 5:

Verify Green's Theorem by evaluating both integrals

$$\int_C y^2 dx + x^2 dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

for the given path.

C: boundary of the region lying between the graphs of $y = x$ and $y = x^2/4$



$$2. \mathbf{r}(t) = \begin{cases} ti + t^2/4\mathbf{j}, & 0 \leq t \leq 4 \\ (8-t)\mathbf{i} + (8-t)\mathbf{j}, & 4 \leq t \leq 8 \end{cases}$$

$$3. \int_C y^2 dx + x^2 dy = \int_0^4 \left[\frac{t^4}{16}(dt) + t^2 \left(\frac{t}{2} dt \right) \right] + \int_4^8 [(8-t)^2(-dt) + (8-t)^2(-dt)]$$

$$4. = \int_0^4 \left[\frac{t^4}{16} + \frac{t^3}{2} \right] dt + \int_4^8 -2(8-t)^2 dt$$

$$5. = \frac{224}{5} - \frac{128}{3}$$

$$6. = \frac{32}{15}$$

7. By Green's Theorem,

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \int_0^4 \int_{x^2/4}^x (2x - 2y) dy dx$$

$$8. = \int_0^4 \left(x^2 - \frac{x^3}{2} + \frac{x^4}{16} \right) dx$$

$$9. = \frac{32}{15}$$

Question : 6

Use Green's Theorem to evaluate the integral

$$\int_C (y - x) dx + (2x - y) dy$$

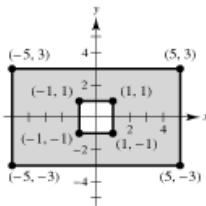
for the given path.

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1.$$

C: boundary of the region lying inside the rectangle bounded by $x = -5$, $x = 5$, $y = -3$, and $y = 3$, and outside the square bounded by $x = -1$, $x = 1$, $y = -1$, and $y = 1$

1. From the accompanying figure, we see that R is the shaded region.

2.



3. Thus, Green's Theorem yields

$$4. \int_C (y - x) dx + (2x - y) dy = \iint_R 1 dA$$

$$5. = \text{Area of } R$$

$$6. = 6(10) - 2(2)$$

$$7. = 56.$$

Question : 7

Evaluate $\iint_S xy dS$.

S: $z = 6 - x - 2y$, first octant

$$1. \frac{\partial z}{\partial x} = -1$$

$$2. \frac{\partial z}{\partial y} = -2$$

$$3. \iint_S xy dS = \int_0^6 \int_0^{3-(x/2)} xy \sqrt{1 + (-1)^2 + (-2)^2} dy dx$$

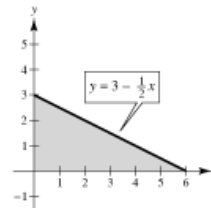
$$4. = \sqrt{6} \int_0^6 \left[\frac{xy^2}{2} \right]_0^{3-(x/2)} dx$$

$$5. = \frac{\sqrt{6}}{2} \int_0^6 x \left(9 - 3x + \frac{1}{4}x^2 \right) dx$$

$$6. = \frac{\sqrt{6}}{2} \left[\frac{9x^2}{2} - x^3 + \frac{x^4}{16} \right]_0^6$$

$$7. = \frac{27\sqrt{6}}{2}$$

8.



Question : 8

Evaluate $\iint_S (x - 2y + z) dS$.

$S: z = 4 - x, 0 \leq x \leq 4, 0 \leq y \leq 4$

1. $\frac{\partial z}{\partial x} = -1$

2. $\frac{\partial z}{\partial y} = 0$

3. $\iint_S (x - 2y + z) dS = \int_0^4 \int_0^4 (x - 2y + 4 - x) \sqrt{1 + (-1)^2 + (0)^2} dy dx$

4. $= \sqrt{2} \int_0^4 \int_0^4 (4 - 2y) dy dx$

5. $= 0$

Question : 9

Find the flux of \mathbf{F} through S ,

$$\iint_S \mathbf{F} \cdot \mathbf{N} dS$$

where \mathbf{N} is the upward unit normal vector to S .

$$\mathbf{F}(x, y, z) = 3z\mathbf{i} - 4y\mathbf{j} + y\mathbf{k}$$

$S: x + y + z = 1$, first octant

1. $G(x, y, z) = x + y + z - 1$

2. $\nabla G(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

3. $\iint_S \mathbf{F} \cdot \mathbf{N} dS = \iint_R \mathbf{F} \cdot \nabla G dA$

4. $= \int_0^1 \int_0^{1-x} (3z - 4 + y) dy dx$

5. $= \int_0^1 \int_0^{1-x} [3(1 - x - y) - 4 + y] dy dx$

6. $= \int_0^1 \int_0^{1-x} (-1 - 3x - 2y) dy dx$

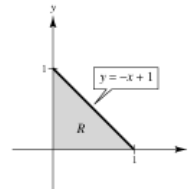
7. $= \int_0^1 \left[-y - 3xy - y^2 \right]_0^{1-x} dx$

8. $= - \int_0^1 [(1 - x) + 3x(1 - x) + (1 - x)^2] dx$

9. $= - \int_0^1 (2 - 2x^2) dx$

10. $= -\frac{4}{3}$

11.



Question : 10

Use the Divergence Theorem to evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, dS$$

and find the outward flux of \mathbf{F} through the surface of the solid bounded by the graphs of the equations. Use a computer algebra system to verify your results.

$$\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$$

$$S: x = 0, x = a, y = 0, y = a, z = 0, z = a$$

1. Since $\text{div } \mathbf{F} = 2x + 2y + 2z$, we have

$$2. \iiint_Q \text{div } \mathbf{F} \, dV = \int_0^a \int_0^a \int_0^a (2x + 2y + 2z) \, dz \, dy \, dx$$

$$3. = \int_0^a \int_0^a (2ax + 2ay + a^2) \, dy \, dx$$

$$4. = \int_0^a (2a^2x + 2a^3) \, dx$$

$$5. = \left[a^2x^2 + 2a^3x \right]_0^a$$

$$6. = 3a^4.$$

Question : 11

Verify Stokes's Theorem by evaluating

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot d\mathbf{r}$$

as a line integral and as a double integral.

$$\mathbf{F}(x, y, z) = (-y + z)\mathbf{i} + (x - z)\mathbf{j} + (x - y)\mathbf{k}$$

$$S: z = \sqrt{1 - x^2 - y^2}$$

1. In this case, $M = -y + z$,

2. $N = x - z$,

3. $P = x - y$ and

4. C is the circle $x^2 + y^2 = 1$,

5. $z = 0$,

6. $dz = 0$.

$$7. \text{Line Integral: } \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (-y + z) \, dx + (x - z) \, dy + (x - y) \, dz$$

$$8. = \int_C -y \, dx + x \, dy$$

9. Letting $x = \cos t$,

10. $y = \sin t$, we have

11. $dx = -\sin t \, dt$,

12. $dy = \cos t \, dt$ and

$$13. \int_C -y \, dx + x \, dy = \int_0^{2\pi} (\sin^2 t + \cos^2 t) \, dt$$

$$14. = 2\pi.$$

15. **Double Integral:** Consider $F(x, y, z) = x^2 + y^2 + z^2 - 1$. Then

$$16. \mathbf{N} = \frac{\nabla F}{\|\nabla F\|}$$

$$17. = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}}$$

$$18. = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

19. Since $z^2 = 1 - x^2 - y^2$,

$$20. z_x = \frac{-2x}{2z}$$

$$21. = \frac{-x}{z}, \text{ and}$$

$$22. z_y = \frac{-y}{z},$$

$$23. dS = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} \, dA$$

$$24. = \frac{1}{z} \, dA.$$

25. Now, since $\text{curl } \mathbf{F} = 2\mathbf{k}$, we have

$$26. \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS = \iint_R 2z \left(\frac{1}{z} \right) \, dA$$

$$27. = \iint_R 2 \, dA$$

$$28. = 2(\text{Area of circle of radius 1})$$

$$29. = 2\pi.$$

